The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before December 31, 2021. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall 2021 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 870-880.

Problem 870. *Propsosed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*

Find all positive integers x, y such that $x^3 - y^3 - xy = 113.$

Problem 871. *Proposed by Seán Stewart, Bomaderry, NSW, Australia.*

Evaluate $\int_0^{\pi/2} cosecx\ log^3\left(\frac{1+\cos x+\sin x}{1+\cos x-\sin x}\right)\, dx$

Problem 872. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Let x, y, z ε $(0,1)$ with $xy + yz + zx = 1$. Prove that

$$4(x+y+z) \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 9xyz.$$

Problem 873. *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Prove that $a^2tan^kx + b^2sin^kx > 2ab\ x^k$ for all a,b > 0, x ε $(0,\frac{\pi}{2})$, and positive integer k.

Problem 874. *Proposed by Abhijit Bhattacharjee, ex Msc student in BHU, India.*

Prove that the equation $a^2 + (a + n\}^2 = b^2$ with a, b, $n \in \mathbb{N}$ has infinitely many solutions for each n.

Problem 875. *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania*.

Calculate the following integral:

$$\int_0^\infty \frac{\arctan(x)}{\sqrt{2x^4 + x^2 + 2}}\, dx$$

Problem 876. *Proposed by Ankush Kumar Parcha (student), Indira Gandhi National Open University, New Delhi, India and Toyesh Prakash Sharma (student), St. C.F Andrews School, Agra, India.*

If $x = \sum_{n=1}^{\infty}\left(x^{2n} + \frac{1}{x^{2n}}\right)$ and $y = \sum_{n=0}^{\infty} \frac{1 + x^{2n+1}}{x^n}$, compute the value of x^y .

Problem 877. *Proposed by Angel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain.*

Let x be a real number. For any positive integer n, find closed forms for the following sums:

$$S_n^e = \sum_{k\,even} \binom{n+1}{k} x^k, \quad S_n^o = \sum_{k\,odd} \binom{n+1}{k} x^k$$

$$F_n^e = \sum_{k\,even} \binom{n+1}{k} F_k, \quad F_n^o = \sum_{k\,odd} \binom{n+1}{k} F_k$$

$$L_n^e = \sum_{k\,even} \binom{n+1}{k} L_k, \quad L_n^o = \sum_{k\,odd} \binom{n+1}{k} L_k$$

Where F_n and L_n are respectively the nth Fibonacci and Lucas numbers defined both by the recurrence relation $u_{n+2} = u_{n+1} + u_n$ with initial values $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$.

Problem 878. *Proposed by Mihaly Bencze, Brasov, Romania.*

Let F_n and L_n be respectively the *n*th Fibonacci and Lucas numbers as defined above. Prove the following two inequalities:

1)
$$\prod_{k=1}^{n}\left(\frac{2F_n}{F_k}-1\right) \geq \left(\frac{2nF_n}{F_{n+2}-1}-1\right)^n$$

2)
$$\prod_{k=1}^{n}\left(\frac{2L_n}{L_k}-1\right) \geq \left(\frac{2nL_n}{L_{n+2}-1}-1\right)^n$$

Problem 879. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let $a > b > 0$. Evaluate $\int_0^{\pi} \frac{\sin^n x}{(a+b\cos x)^{n+1}}\,dx$ for $n = 0, 1, 2, \ldots$

Problem 880. *Proposed by Dorin Marghidanu, Colegiul National 'A.I. Cuza', Corabia, Romania.*

Let $n \geq a_k$, $b_k > 0$ with n and p integers ≥ 2. Prove that

$$\frac{1}{\sqrt[p]{a_1 a_2 \ldots a_n}} + \frac{1}{\sqrt[p]{b_1 b_2 \ldots b_n}} \geq \frac{2\sqrt[p]{2^n}}{\sqrt[p]{(a_1+b_1)(a_2+b_2)\ldots(a_n+b_n)}}$$