The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before July 31, 2023. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Spring 2023 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 901-910.

Problem 901. *Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*

Suppose for some integer $k \geq 2$ that $a_1 < a_2 < \ldots < a_k$ are positive integers, and that A is their least common multiple. Prove that

$$a_1a_2 + a_2a_3 + \cdots + a_{k-1}a_k + a_ka_1 \leq A^2.$$

Problem 902. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Without a computer, find $\Omega = \int_0^{\pi/30} \frac{\sin 5x \, sin7x \, sin8x}{\cos 2x \cos 3x \cos 5x \cos 10x}\, dx$

Problem 903. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Solve for complex numbers:

$$\begin{vmatrix} x^2 & x^2+3x & x^2+6x+9 \\ x^2+2x+1 & x^2+5x+4 & x^2+8x+16 \\ x^4+2x^2+1 & x^4+3x^2+2 & x^4+4x^2+4 \end{vmatrix} = 0$$

Problem 904. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

Find the nth term of the sequence $(a_n)_{n\geq 0}$ defined recursively as follows:

$a_0 = 0,\ a_1 = 1,\ a_2 = 0,$

$\forall n \geq 3 :\ a_n = \sum_{k=1}^{n-1}(n-k)(n-k-4)a_k$

Problem 905. *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the following integral without a computer:

$$\int_1^\infty \frac{x\sqrt{x}\ln x}{(x+1)(x^2+1)}\,dx$$

Problem 906. *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

If $\lambda \geq 1$ and ABC is a triangle, prove that $\sum\left(tan\frac{A}{4}\right)^{\lambda} \geq 3\left(2-\sqrt{3}\right)^{\lambda}$.

Problem 907. *Proposed by Toyesh Prakash Sharma (student), Agra College, Agra, India.*

If $a,b,c > 0$, then show that

$$\left(\frac{a}{b+c}\right)^{\frac{a}{b+c}} + \left(\frac{b}{c+a}\right)^{\frac{b}{c+a}} + \left(\frac{c}{a+b}\right)^{\frac{c}{a+b}} \geq 3^{\frac{2}{3}}.$$

Problem 908. *Proposed by Raluca Maria Caraion and Forică Anastase, "Alexandru Odobescu" High School, Lehliu-Gară, Călăraşi, Romania.*

If $a,b,c > 0$, then show that

$$\prod\frac{(1+ab)(1+ac)}{1+a\sqrt{bc}} \geq \left(1+\sqrt[3]{a^2b^2c^2}\right)^3$$

Problem 909. *Proposed by Seán Stewart, King Abdullah University of Science and Technology, Saudi Arabia.*

If $m > 1$, without a computer evaluate

$$\int_0^{\pi/2}\frac{\cot\left(\frac{x}{2}\right)\sec x\log(\cos x)}{\sqrt[m]{\sec x-1}}\,dx$$

Problem 910. *Proposed by the editor.*

Prove that the sum of the last two digits of 2^n is never 17.