The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before October 1, 2015. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall 2015 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 749-759

Problem 749. *Proposed by Tom Moore, Professor Emeritus, Bridgewater State University, Bridgewater, MA.*

The pentagonal numbers are 1, 5, 12, 22, … and are given by $P_n = \dfrac{n(3n-1)}{2}, \forall n \geq 1$.

Prove that every positive even power of 2 is expressible as the difference of two pentagonal numbers.

Problem 750. *Proposed by Tom Moore, Professor Emeritus, Bridgewater State University, Bridgewater, MA.*

It is fairly well known that if (a,b,c) is a primitive Pythagorean triple (PPT), then the product abc is divisible by 60. Find infinitely many PPTs (a,b,c) such that abc is divisible by 120.

Problem 751. *Proposed by Iuliana Trasca, Olt, Romania*.

Prove that if $a,b,c > 0$, then $a^{11} + b^{11} + c^{11} \geq a^4b^4c^4\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$

Problem 752. *Proposed by Iuliana Trasca, Olt, Romania*.

Prove that if $x,y,z > 0$, then $\frac{2x+2y+4z}{4x+4y+3z}+\frac{2x+4y+2z}{4x+3y+4z}+\frac{4x+2y+2z}{3x+4y+4z}\geq\frac{24}{11}$

Problem 753. *Proposed by D.M. Batinetu-Giurgiu, "Matie Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania*.

If $a,b,c > 0$ and $m\geq 0$, then prove that $(a+b+c)^{2m}\left(\frac{1}{(ab)^m}+\frac{1}{(bc)^m}+\frac{1}{(ca)^m}\right)\geq 3^{2m+1}$

Problem 754. *Proposed by D.M. Batinetu-Giurgiu, "Matie Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania*.

Prove that if $a,b > 0$, then $\frac{4}{\sqrt{ab(a+b)}}\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{ab}$

Problem 755. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Sara has a total number of 26 paper bills in denominations of \$1, \$5, \$10 and \$20 in her purse. The number of \$5 bills is 4 times the number of \$10 bills, and the number of \$1 bills is 1 less than twice the number of \$5 bills. She remembers that the total amount is less than \$100 and more than \$90. Furthermore, she remembers that the total amount is an odd number, but it is not 91 or 99. How much money does she have and what is the number of each denomination?

Problem 756. *Proposed by Jose Luis Diaz-Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain*.

Find all triples of nonzero real numbers x,y,z such that

$$36\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)=1$$

$$36\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right)=49$$

$$36^2\left(\frac{1}{x^2y^2}+\frac{1}{y^2z^2}+\frac{1}{z^2x^2}\right)=49$$

Problem 757. *Proposed by Jose Luis Diaz-Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain*.

Let a, b be two complex numbers lying on the circle $|z| = 1$. Prove that

$$\left(\frac{a+b}{1+ab}\right)^2+\left(\frac{a-b}{1-ab}\right)^2 \geq 1$$

Problem 758. *Proposed by the editor.*

Prove that the sequence 11231, 1012301, 100123001, 10001230001 ... (i.e, each number starts and ends with 1 and has $k \geq 0$ zeroes on either side of 123) has an infinite subsequence of all composite numbers.

Problem 759. *Proposed by Marcel Chirita, Bucharest, Romania.*

Show how to evaluate the definite integral $\int_0^1 \frac{(x-x^2)\arctan x}{(1+x)(1+x^2)}dx$