The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before March 15, 2018. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Spring 2018 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 798-807

Problem 798. *Proposed by the editor*.

In 2002, Britney Gallivan (high school junior) found a formula for paper folding and managed to do 12 folds of a long sheet of toilet paper. She found that

$$L = \frac{\pi t}{6}\left(2^n + 4\right)\left(2^n - 1\right)$$

where t represents the thickness of the material to be folded, L is the length of the paper to be folded and n is the number of folds desired (in only one direction). Suppose you tape together sheets of standard 8.5” x 11” copier paper (thickness .0035”) end to end, how many sheets would be needed to be able to fold the long taped sheet 14 times?

Problem 799. *Proposed by Daniel Sitaru, “Theodor Costescu” National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a,b,c \in (0,2]$ then

$$3\sqrt{2} \le \sum \frac{b(\sqrt{a}+\sqrt{2-a}}{c} \le 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$$

Problem 800. *Proposed by Daniel Sitaru, “Theodor Costescu” National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a \in \mathbb{R}$, then

$$\int_{a+3}^{a+5} \ln(1+e^x)dx + \int_{a+6}^{a+8} \ln(1+e^x)dx \leq \int_{a}^{a+2} \ln(1+e^x)dx + \int_{a+9}^{a+11} \ln(1+e^x)dx$$

.

Problem 801. *Proposed by Jose Luis Diaz-Barrero, Barcelona Tech-UPC, Barcelona, Spain.*

Compute

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} \frac{k^2+n^2}{1+2\sqrt{\dfrac{k^2+n^2+n^3}{n^3}}}$$

Problem 802. *Proposed by Jose Luis Diaz-Barrero, Barcelona Tech-UPC, Barcelona, Spain.*

Let $n \geq 1$ be an integer. Prove that

$$\sqrt[n]{\prod_{k=1}^{n} F_{k+1}} \geq \frac{1}{2}\left(\sqrt[n]{\prod_{k=1}^{n} F_k} + \sqrt[n]{\prod_{k=1}^{n} L_k} \right)$$

where F_n and L_n are the nth Fibonacci and Lucas numbers defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ and by $L_1 = 1$, $L_2 = 3$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

Problem 803. *Proposed by Ovidiu Furdui and Alina Sintamarian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*.

Calculate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{H_{n+m}}{n(n+m)^2}$$

where $H_n = 1 + ½ + \ldots + 1/n$ denotes the nth harmonic number.

Problem 804. *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Compute the following limit

$$\lim_{n\to\infty} \frac{\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}{\sqrt[n+1]{(2n+1)!!}}$$

Problem 805. *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Let $z_k = x_k + i\, y_k$ be a complex number where $k \in \{ 1,2, \ldots, n \}$. Prove that

$$\sum_{k=1}^{n} \sqrt{{x_k}^4 + {y_{n-k+1}}^4} \geq \frac{\sqrt{2}}{2} \sum_{k=1}^{n} | z_k |^2$$

Problem 806. *Proposed by Marius Dragan, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

If $a_1, a_2, \ldots, a_n > 0$ such that $\sum_{k=1}^{n} a_k = 1$, then prove that

$$\left(1+1/a_2\right)^{na_1^2} \left(1+1/a_3\right)^{na_2^2} \ldots \left(1+1/a_n\right)^{na_{n-1}^2} \left(1+1/a_1\right)^{na_n^2} \geq n+1$$

Problem 807. *Proposed by Titu Zvonaru, Comanesti, Romania.*

If A, B, and C are the angles of a triangle and $\forall = A/2$, $\exists = B/2$, $(= C/2$, prove that

$$\sqrt{6(1+\cos A \cos B \cos C) - 2\sin\alpha \sin\beta \sin\gamma (1 - 8\sin\alpha \sin\beta \sin\gamma)} \geq 4\cos\alpha \cos\beta \cos\gamma$$

SOLUTIONS TO PROBLEMS 780-788

Problem 780. *Proposed by Daniel Sitaru, Colegiul National Economic College Theodor Costescu, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a,b,c \in [1,\infty)$, then $ab + bc + ca \geq 3 + 2 \ln(a^b b^c c^a)$.

Solution *by Richdad Phuc, University of Sciences, Hanoi, Vietnam.*

We have $\text{LHS} - \text{RHS} = b(a - 2 \ln a) + c(b - 2 \ln b) + a(c - 2 \ln c) - 3$ or

$$\text{LHS} - \text{RHS} = (b/a)a(a - 2 \ln a) + (c/b)b(b - 2 \ln b) + (a/c)c(c - 2 \ln c) - 3$$

Let $f(x) = x(x - 2 \ln x)$ for $x \geq 1$. The derivative is $f'(x) = 2x - 2 \ln x - 2$ and $f''(x) = 2 - 2/x \geq 0$ for all $x \geq 1$ □ $f'(x) \geq f'(1) = 0$ for all $x \geq 1$.

This means $f(x)$ is strictly increasing on $[1, \infty)$. Then $f(x) \geq f(1)$ for all $x \geq 1$. Hence $a(a - 2 \ln a) \geq 1$, $b(b - 2 \ln b) \geq 1$, $c(c - 2 \ln c) \geq 1$. Then

$$\text{LHS} - \text{RHS} \geq (b/a) + (c/b) + (a/c) - 3$$ which is ≥ 0 by the AM-GM inequality.

Equality holds if $a = b = c = 1$.

Also solved by *Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Henry Ricardo, New York Math Circle, NY; and the proposer.*

Problem 781. *Proposed by Daniel Sitaru, Colegiul National Economic College Theodor Costescu, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a,b,c \in (0,\infty)$, then

$$\sum a\sqrt{(b^4+c^4)/2} \le a^2(b+c)+b^2(a+c)+c^2(a+b)-3abc\,.$$

Solution *by the proposer.*

We prove that if $x,y \in (0,\infty)$, then $\quad x+y-\sqrt{xy} \ge \sqrt{\dfrac{x^2+y^2}{2}} \quad (*)$.

We denote $u=\sqrt{\dfrac{x^2+y^2}{2}}$ which means $2u^2 = x^2 + y^2$ and let $v=\sqrt{xy}$ □ $v^2 = xy$

With these notations, we have $2u^2 + 2v^2 = x^2 + 2xy + y^2 = (x+y)^2$

We can rewrite (*) as $x + y - v \ge u$ or $(x + y)^2 \ge (u + v)^2$

$\Leftrightarrow 2u^2 + 2v^2 \ge (u + v)^2$

$\Leftrightarrow 2u^2 + 2v^2 - u^2 - v^2 - 2uv \ge 0$

$\Leftrightarrow (u - v)^2 \ge 0$

Now replace x with x/y and y with y/x in (*) to get

$$\frac{x}{y}+\frac{y}{x} \ge \sqrt{\frac{(x/y)^2+(y/x)^2}{2}}+\sqrt{\frac{x}{y}\cdot\frac{y}{x}}$$

$$\Leftrightarrow \frac{x^2+y^2}{xy} \ge \frac{1}{xy}\sqrt{\frac{x^4+y^4}{2}}+1$$

$$\Leftrightarrow x^2+y^2 \ge \sqrt{\frac{x^4+y^4}{2}}+xy$$

For $x = a$ and $y = b$ and multiplying by c we have

$$a^2c+b^2c \ge c\sqrt{\frac{a^4+b^4}{2}}+abc$$

Analogously,

$$b^2a+c^2a \ge a\sqrt{\frac{b^4+c^4}{2}}+abc$$

and

$$c^2b+a^2b \ge b\sqrt{\frac{c^4+a^4}{2}}+abc$$

Adding the last three inequalities gives the desired result.

Also solved by Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Soumitra Moukherjee, Scottish Church College, Chandar Nagore, India; Ravi Prakash, Oxford University Press, New Delhi, India.

Problem 782. *Proposed by Jose Luis Diaz-Barrero, Barcelona Tech-UPC, Barcelona, Spain.*

Let a, b, c be the lengths of the sides of triangle ABC and m_a, m_b, and m_c the lengths of its medians. Prove that

$$\frac{2^{m_a}+2^{m_b}+2^{m_c}}{2^a+2^b+2^c}<1.$$

Solution *by Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Since $m_c<\dfrac{a+b}{2}$, by the AM-GM inequality we have

$$2^a+2^b\geq 2\sqrt{2^a2^b}=2\cdot 2^{\frac{a+b}{2}}>2\cdot 2^{m_{c\partial}}$$

Writing the other two similar inequalities and adding all three gives the desired result.

Also solved by Madison Estabrook, Missouri State University, Springfield, MO; Rovsen Pirkulyev, Baku State University, Sumgait, Azerbaidjian; and the proposer.

Problem 783. *Proposed by Jose Luis Diaz-Barrero, Barcelona Tech-UPC, Barcelona, Spain.*

Find all real solutions of the following system of equations

$$x^3 + 2x + y = 9 + 3x^2$$
$$3y^2 + 6y + z = 21 + 9y^2$$
$$5z^3 + 10z + x = 33 + 15z^2.$$

Solution *by the proposer.*

We can rewrite the system as

$$3 – y = x^3 - 3x^2 + 2x – 6$$
$$3 – z = 3(y^3 – 3y^2 + 2y - 6)$$
$$3 – x = 5(z^3 – 3z^2 +2z – 6)$$

Since $t^3 – 3t^2 +2\,t - 6 = (t – 3)(t^2 +2)$, we have

$$3 – y = (x – 3)(x^2 +2)$$
$$3 – z = 3(y – 3)(y^2 +2)$$
$$3 – x =5(z – 3)(z^2 +2).$$

Multiplying these together gives

$$-(x-3)(y-3)(z-3) = 15(x-3)(y-3)(z-3)(x^2 +2)(y^2 +2)(z^2 +2).$$

From this we get

$$0 = (x-3)(y-3)(z-3)(15(x^2 +2)(y^2 +2)(z^2 +2) + 1)$$

Since the last factor above is positive, either $x = 3$, $y = 3$ or $z = 3$.
If we assume that $x = 3$, the first equation says $y = 3$. Substituting this into the second equation implies that z=3. The same occurs for starting with y=3 or z=3. So $x = y = z = 3$ is the only real solution.

Also solved by Soumava Chakraborty, Softweb Technologies, Kolkota, India.

Problem 784. *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Prove that in any triangle ABC with $BC = a$, $CA = b$, $AB = c$ and area F, the following inequalities are true.

$$(b^2+c^2)\sin\frac{A}{2}+(c^2+a^2)\sin\frac{B}{2}+(a^2+b^2)\sin\frac{C}{2}\geq 4\sqrt{3}F,$$

$$ab(1+\sin^2\frac{C}{2})+bc(1+\sin^2\frac{A}{2})+ca(1+\sin^2\frac{B}{2})\geq 4\sqrt{3}F.$$

Solution *by Ioan Viorel Codreanu, Satulung, Maramures. Romania.*
We have $(b^2+c^2)\sin\frac{A}{2}\geq 2bc\sin\frac{A}{2}=\frac{bc\sin A}{\cos\frac{A}{2}}=\frac{2F}{\cos\frac{A}{2}}=2F\sec\frac{A}{2}$

Similarly, $(c^2+a^2)\sin\frac{B}{2}\geq 2F\sec\frac{B}{2}$ and $(a^2+b^2)\sin\frac{C}{2}\geq 2F\sec\frac{C}{2}$

Then $\sum(b^2+c^2)\sin\frac{A}{2}\geq 2F\sum\sec\frac{A}{2}$.

Using Jensen's Inequality and that f(x) = sec x on (0, π/2) is a convex function, we get
$\sum\sec\frac{A}{2}\geq 3\sec\frac{\sum A}{6}=2\sqrt{3}.$ Thus $\sum(b^2+c^2)\sin\frac{A}{2}\geq 4\sqrt{3}F.$

Next $ab(1+\sin^2\frac{C}{2})\geq 2ab\sin\frac{C}{2}=\frac{ab\sin C}{\cos\frac{C}{2}}=2F\sec\frac{C}{2}.$

Similarly, $bc(1+\sin^2\frac{A}{2})\geq 2F\sec\frac{A}{2}$ and $ca(1+\sin^2\frac{B}{2})\geq 2F\sec\frac{B}{2}.$

Then $\sum ab(1+\sin^2\frac{C}{2})\geq 2F\sum\sec\frac{A}{2}\geq 4\sqrt{3}F.$

Also solved by Soumava Chakraborty, Softweb Technologies, Kolkota, India; Ravi Prakash, Oxford University Press, New Delhi, India; Soumitra Moukherjee, Scottish Church College, Chandar Nagore, India; and the proposer.

Problem 785. *Proposed by Iuliana Trasca, Scornicesti, Romania*.

Show that $x, y, z > 0$, then

$$\frac{x^6z^3+y^6x^3+z^6y^3}{x^2y^2z^2} \geq \frac{x^3+y^3+z^3+3xyz}{2} \ .$$

Solution *by Soumava Chakraborty, Softweb Technologies, Kolkota, India.*
The inequality is equivalent to
$2(x^6z^3 + y^6x^3 + z^6y^3) \geq x^5y^2z^2 + y^5z^2x^2 + z^5x^2y^2 + 3x^3y^3z^3$.
Using the AM-GM inequality, we have

$$x^6z^3 + x^6z^3 + x^3y^6 \geq 3x^5y^2z^2$$
$$y^6x^3 + y^6x^3 + y^3z^6 \geq 3y^5z^2x^2$$
$$z^6y^3 + z^6y^3 + z^3x^6 \geq 3z^5x^2y^2$$

Adding these gives

$$3(x^6z^3 + y^6x^3 + z^6y^3) \geq 3(x^5y^2z^2 + y^5z^2x^2 + z^5x^2y^2).$$

Dividing by 3 says

$$x^6z^3 + y^6x^3 + z^6y^3 \geq x^5y^2z^2 + y^5z^2x^2 + z^5x^2y^2$$

The AM-GM inequality also says

$$x^6z^3 + y^6x^3 + z^6y^3 \geq 3x^3y^3z^3.$$

Summing the previous two inequalities gives the inequality that is equivalent to the one of the problem.

Also solved by Soumitra Moukherjee, Scottish Church College, Chandar Nagore, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Titu Zvonaru, Comanesti, Romania; and the proposer.

Problem 786. *Proposed by Thomas Chu, Macomb, Illinois*.

Prove that if $x,y,z > 1$, then

$$(x^2+y^2+z^2)(x+y+z)+x^3+y^3+z^3 > 4xy+4xz+4yz \ .$$

Solution *by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*.
By changing variables $x = 1+a$, $y = 1+b$ and $z = 1+c$, the problem reads as:
Prove that if $a,b,c > 0$, then

$$\left(\sum(1+a)^2\right)\left(3+\sum a\right)+\sum(1+a)^3 > 4(1+a)(1+b)+4(1+b)(1+c)+4(1+c)(1+a)$$

Expanding the right-hand side and left-hand sides, we get

$$\text{LHS} = 12 + 12(a+b+c) + 4(ab+ac+bc) + 8\sum a^2 + \sum a^2b + 2\sum a^3$$
$$\text{RHS} = 12 + 8(a+b+c) + + 4(ab+ac+bc)$$

We can clearly see that the LHS > RHS when $a,b,c > 0$.

Also solved by Anas Adlany (student), Omar Ben Abdelaziz University, El Jadida, Morroco; Myagmarsuren Yadamsuren, Ulanbataar University, Ulanbataar, Mongolia; Soumava Chakraborty, Softweb Technologies, Kolkota, India; and the proposer.

Problem 787. *Proposed by the editor*.

Mike buys some pants and shorts at the Great Pants Store. Mike buys shorts that cost \$11 each and pants that cost \$14 each. His total before taxes is \$283. How many shorts and how many pants did Mike buy?

Solution *by Robert Bailey (former KME national President 2001-2005), Niagara University, NY*.
Let x = number of shorts and y = number of pants. We have $11x + 14y = 283$ which is a linear Diophantine equation in two variables. Then $14y = 283 – 11x$ which is equivalent to $14y \equiv 283 \pmod{11}$ or $3y \equiv 8 \pmod{11}$ or $3y \equiv -3 \pmod{11}$. Since 3 is relatively prime to 11, we get $y \equiv -1 \pmod{11}$. This means $y = 10, 21, 32, \ldots$. The only value for y that causes x to be positive in the equation $11x + 14y = 283$ is $y = 10$ in which case $x = 13$.

Also solved by Michael Bhujel, Bobbie Legg, Katie Tyson (students), and Bill Yankosky, North Carolina Wesleyan College, Rocky Mount, NC; Jeremiah Bartz, University of North Dakota, Grand Forks, ND; and the proposer.

Problem 788. *Proposed by George Heineman, Worcester Polytechnic Institute, Worcester, MA.*

A Sujiken™ puzzle has a triangular grid of cells containing digits from 1 to 9. You must place a digit in each of the empty cells with the constraint that no digit can repeat in any row, column, or diagonal. Additionally, no digit can repeat in the 3x3 large squares with thick borders or the three triangular regions with thick borders. The puzzle below is of intermediate difficulty.

			8	5	9	4		
			6			3		
			3				1	9
			1	9				7
					6			1
					5	8	4	6

Solution

6	1	3	8	5	9	4	7	2
	9	7	6	1	2	3	5	8
		2	3	7	4	6	1	9
			1	9	8	5	3	7
				4	6	2	9	1
					5	8	4	6
						7	2	5
							8	4
								3

Solved by Jamie Farrar, Destinee Fisher, Nicole Kettle, Courtney Lush (students), Ed Wilson (retired faculty), Eastern Kentucky University, Richmond, KY; Jeremiah Bartz, University of North Dakota, Grand Forks, ND; Katie Tyson (student), Gail Stafford, Carol Lawrence, Bill Yankosky, North Carolina Wesleyan College, Rocky Mount, NC.