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On Health Care: Making an Informed Decision

Fred N. Hollingshead, *student*

KS Delta

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Presented at the 2005 National Convention and awarded "top four" status by the Awards Committee.

1. Purpose

Often, in today's work environment, employers offer their associates two or more health insurance plans. Most people select one plan over another primarily for economic reasons, yet they do not conduct a true cost analysis. Foregoing any mathematical examination, people estimate (some accurately, and some not so) their costs and without much more thought, proceed with their choice, and hope for the best. With both insurance and medical costs rising at an alarming rate, choosing the most economical health insurance plan represents a significant real-world problem for many people. This paper will discuss the methods employed to examine a specific case involving the choice of two insurance plans and an available tax savings option. Before beginning, the reader should first recognize the two objectives of this discussion. First, to determine the more economical plan for the employee, we must develop an algorithm to perform the cost analysis, and second, a closer investigation of the tax savings option will be completed to explain the benefits of this opportunity. Waterloo Maple's mathematics software *Maple version 9* will be used to aid the analysis both numerically as well as graphically.

2. Assumptions/Terminology

Washburn University in Topeka, KS offers their faculty and staff two plan choices, the *Base Plan* and the *Buy-up Plan*. Table 1 below outlines some of the major features of each plan, and with a glance, the Buy-up Plan clearly has "better" coverage; however, this improved coverage comes at a price; the Buy-up Plan's considerably higher annual *premiums*[1] (Appendix A contains a complete table showing premium costs for both plans), hence the conundrum employees face when selecting their coverage. We shall develop a model which will compare only differences in the coverage, though the reader should be aware some major features of the plans are identical and consequently not included in Table 1 as well as ignored in our model.

From Table 1, we see small differences in *co-pay*[2] and *prescription*[3] benefits. Heavy utilization of these benefits could affect the decision process, but initially, we shall ignore these differences. Additionally, we consider only full-time employment. The initial model will focus completely on the *deductibility*[4] and *coinsurance*[5] features of the Base and Buy-up Plans.

Our cost analysis will necessitate a comparison of actual (after insurance) out-of-pocket expenses the employee would pay with each plan. These expenses will depend entirely upon the estimated amount of annual medical expenses, called qualifying expenses, which would qualify for the deductible and/or coinsurance. The reader must note the employee may have medical expenses not considered as qualified expenses. At Washburn, examples of non-qualifying expenses are doctor co-pays, drug expenses, eye glasses, dental expenses, certain lab fees, etc., which do not qualify for

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[1] **Premiums** are the costs of the insurance paid by the employee, usually with a paycheck deduction.

[2] **Co-pay** is the amount of a doctor's office visit not paid by insurance. Under usual circumstances, including Washburn's plans, co-payments made by the employee to a doctor's office do not count towards any other categories within a plan, nor does the portion paid by the insurance company.

[3] **Prescription** coverage includes any medications prescribed by a doctor. Washburn's plans include three types: **formulary**, which are drugs included on a list provided by the plan provider; **non-formulary**, which are drugs not on the same list and often much more expensive; and **generics**, which are those formulary drugs available under a non-brand name and therefore much less expensive.

[4] **Deductible** is the amount of money which the insured person(s) must pay *before* the insurance begins to cover part or all of any medical costs.

[5] **Coinsurance** is the amount of money which the insured person(s) must pay *after* the deductible has been fulfilled. This is usually a percentage, which varies from plan to plan, of the incurred medical costs and is "capped" at some specified amount. When the cap is reached, the insurance plan then pays for 100% of incurred medical expenses.

deductibility or coinsurance. Finally, the employee accumulates qualified expenses during a fixed time period called the insurance year. Washburn's insurance year differs from the calendar year, instead beginning on November 1 and ending on the following October 31. We begin by analyzing the single coverage expenses.

Table 1: Plan Comparison[6]

	Base Plan
Deductible	$500 Employee $1000 Employee & Dependents
Coinsurance	50% to $1000 Employee to $2000 Employee & Dependents
Co-pay	$20
Prescriptions	$5 Generic/$30 Formulary Brand/ $60 Non-Formulary Brand with oral contraceptives

	Buy-up Plan
Deductible	$250 Employee $500 Employee & Dependents
Coinsurance	20% to $1000 Employee to $2000 Employee & Dependents
Co-pay	$15
Prescriptions	$5 Generic/$25 Formulary Brand/ $50 Non-Formulary Brand with oral contraceptives

[6] These plans are adiministered by Blue Cross/Blue Shield and include other benefits which remain the same for both plans and therefore are negligible in the analysis.

3. Development of the Single Plan Expense Function

Actual out-of-pocket expenses include any medical expenses not covered by the insurance, not including annual premium costs. Let $S_1(x)$ denote the estimated Buy-up Plan out-of-pocket expenses for single employees, where x represents the qualified expenses for the given insurance year. From Table 1 above, the deductible for single employees enrolled in the Buy-up plan is \$250 and the employee pays 20% after the deductible is met until the employee pays an additional \$1000 out-of-pocket. At this point, the insurance plan pays 100% of the expenses incurred. Thus:

$$S_1(x) = x, \text{for } 0 \leq x \leq 250,$$

and after meeting the deductible, the employee pays 20% of the next \$5000 of qualified expenses, meeting the additional \$1000 out-of-pocket requirement. Thus, \$5250 is the next upper bound. So:

$$S_1(x) = 250 + .2(x - 250) = .2x + 200, \text{ for } 250 < x \leq 5250.$$

Note the employee has no additional out-of-pocket expenses after $x = 5250$ as the insurance then pays 100% of all qualifying expenses. Summarizing:

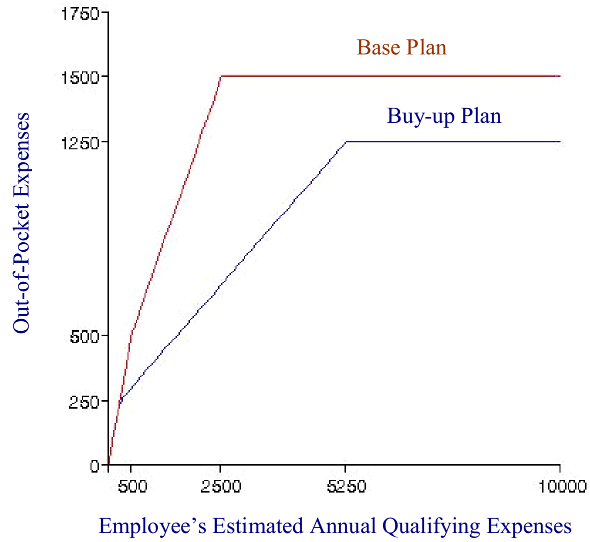
$$S_1(x) = \begin{cases} x & 0 \leq x \leq 250 \\ .2x + 200 & 250 < x \leq 5250 \\ 1250 & x > 5250 \end{cases}.$$

Similarly, $S_2(x)$ will denote the estimated Base plan out-of-pocket expenses for single employees. It follows:

$$S_2(x) = \begin{cases} x & 0 \leq x \leq 500 \\ .5x + 250 & 500 < x \leq 2500 \\ 1500 & x > 2500 \end{cases}.$$

In Figure 1, note $S_1(x)$ and $S_2(x)$ do not include out-of-pocket annual premium costs.

Figure 1: Out-of-Pocket Expense Comparison



From Figure 1, the reader should note the maximum out-of-pocket expenses without premiums for $S_1(x) = \$1250$ (Buy-up Plan) and $S_2(x) = \$1500$ (Base Plan), and the employee incurs more out-of-expense faster with the Base Plan. From Appendix A, the annual premium cost for the Base Plan is covered by Washburn University for single employees; however, the out-of-pocket cost for the Buy-up Plan is \$312 above what Washburn covers. Figure 2 shows the actual out-of-pocket expenses incurred when introducing these added premium costs. The reader should quickly observe the difference between the two figures. With this added expense, the Buy-up Plan is now more economical only for certain values of x. We shall now develop a cost-comparison function to help analyze the plan differences.

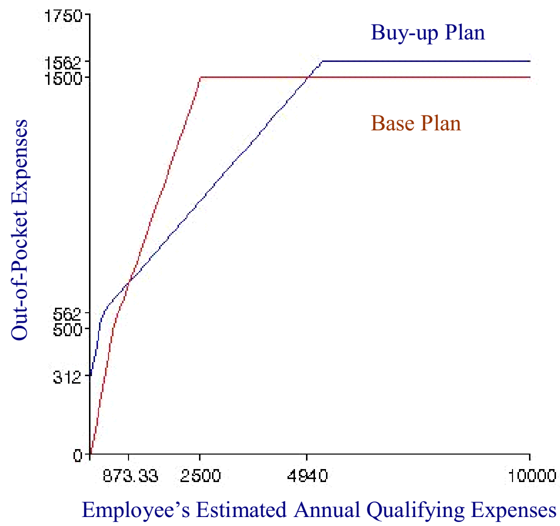
4. Development of the Cost-Comparison Function

Without loss of generality, we shall consider the cost-comparison function, denoted $C_i(x)$ as the difference between the Buy-up and Base Plans' *total*, after insurance, out-of-pocket expenses, including annual premium costs. Thus,

$$C_1(x) = (S_2(x) + p_2) - (S_1(x) + p_1),$$

where p_i is the annual premium costs for each plan, respectively. Arranging the function with like terms grouped together yields:

Figure 2: Out-of-Pocket Expense Comparison with Premiums Included



$$C_1(x) = A + S_2(x) - S_1(x),$$

where $A = p_2 - p_1$, the annual premium difference. Then:

$$C_1(x) = \begin{cases} A + (x - x) & 0 \le x \le 250 \\ A + [(.2x + 200) - (x)] & 250 < x \le 500 \\ A + [(.2x + 200) - (.5x + 250)] & 500 < x \le 2500 \\ A + [(.2x + 200) - (1500)] & 2500 < x \le 5250 \\ A + (1250 - 1500) & x > 5250 \end{cases},$$

so that

$$C_1(x) = \begin{cases} A & 0 \le x \le 250 \\ A - .8x + 200 & 250 < x \le 500 \\ A - .3x - 50 & 500 < x \le 2500 \\ A + .2x - 1300 & 2500 < x \le 5250 \\ A - 250 & x > 5250 \end{cases}.$$

Recall the annual premium cost, p_1, for the Base Plan is \$0, while p_2, the annual premium cost for the Buy-up Plan, is \$312. Substituting A into

$C_1(x)$ results in

$$C_1(x) = \begin{cases} 312 & 0 \le x \le 250 \\ .8x + 512 & 250 < x \le 500 \\ .3x + 262 & 500 < x \le 2500 \\ .2x - 988 & 2500 < x \le 5250 \\ 62 & x > 5250 \end{cases}.$$

Recall we have chosen to ignore prescription drug benefits and doctor co-pay benefits. From Table 1, we see the selection of the Buy-up Plan will save \$5 (or possibly \$10) on certain occasions when the employee uses these benefits. If an employee expects a total of $\$B$ of such savings throughout an insurance year, then we could replace the value of A in the discussion with $A - B$. A, the premium cost penalty for selecting the Buy-up Plan over the Base Plan is reduced appropriately by B. Accordingly, the reader could easily modify the model to include charges from co-pay and prescription drug benefits. In our model, we will continue to assume $B = \$0$ as we proceed.

Figure 3: Buy-up Plan vs. Base Plan

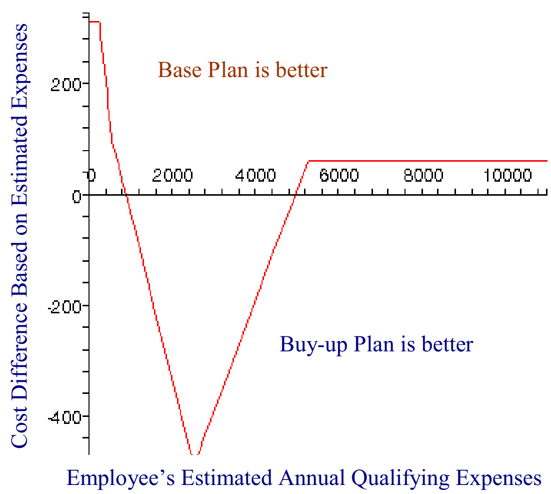


Figure 3, then, shows the resulting graph. As we chose to find the difference of Buy-up expenses minus Base expenses, when $C_1(x) > 0$, the Base Plan is more economical, and when $C_1(x) < 0$, the Buy-up Plan is more economical. Finally, $C_1(x) = 0$ represents the break even points. These break even points can also be seen in Figure 2 where the two graphs intersect. We can easily find these break even points algebraically by setting $C_1(x) = 0$ and solving for x. We then find when an employee's estimated annual qualifying expenses are approximately \$873 or \$4940, it makes little or no difference which plan is chosen. In fact, for employees choosing single coverage, the graph shows the worst case scenario (the cost of choosing the wrong plan) results in at most a loss of \$312 if the Buy-up Plan is chosen and no medical expenses are incurred (note the \$312 comes directly from the annual premium cost) and approximately \$500 if the employee selects the Base Plan and incurs around \$2400 in expenses. Having completed the first analysis, we shall now show similar results for the non-single coverage.

5. Non-Single Results

Non-single employees may choose from three different packages, depending on their individual family situation: Employee + Child(ren), Employee + Spouse, and Employee + Family. The deductibility and coinsurance benefits for all three packages remain the same and are outlined in Table 1.

Following the previous methods, the resulting piecewise functions for non-single coverage are as follows:

$$N_1(x) = \begin{cases} x & 0 \leq x \leq 500 \\ .2x + 400 & 500 < x \leq 10,500 \\ 2500 & x > 10,500 \end{cases},$$

$$N_2 = \begin{cases} x & 0 \leq x \leq 1000 \\ .5x + 500 & 1000 < x \leq 5000 \\ 3000 & x > 5000 \end{cases},$$

$$C_2(x) = \begin{cases} A & 0 \leq x \leq 500 \\ A - .8x + 400 & 500 < x \leq 1000 \\ A - .3x - 100 & 1000 < x \leq 5000 \\ A + .2x - 2600 & 5000 < x \leq 10,500 \\ A - 500 & x > 10,500 \end{cases},$$

where N_1 denotes the estimated Buy-up Plan out-of-pocket expenses for employees electing non-single, $N_2(x)$ denotes the estimated Base Plan out-of-pocket expenses for the same employees, and $C_2(x)$ represents the associated cost-comparison function. The plan's administrators base the progressive premiums, as seen in Appendix A, on an indexed salary schedule. The lower salary tiers pay a smaller percentage of the premium costs than the higher salary tiers. The six-tiered premium costs only apply to the three different non-single packages. As already discussed, employees selecting single coverage pay the same premiums (either free or \$312/year) regardless of salary level, but for non-single plans, eighteen different premium differences (denoted A above) exist. The cost-comparison $C_2(x)$ equation above can be used for all eighteen cases by adjusting the value of A appropriately for each case. A list of the break even points for each of the salary levels and plan coverage can be found in Appendix B.

6. The Tax Savings Modification – Single Plan

In addition to the Base and Buy-up insurance options offered by Washburn University, faculty and staff have an opportunity to invest in a flexible spending account, hereon referred to as *Flex*. The Flex option allows employees to invest pre-tax dollars into an account reserved to pay for various types of medical expenses incurred throughout the year. As before, we will first consider the single coverage case. New assumptions concerning Flex must now be introduced. First, to participate in this option, Flex rules demand employees invest no less than \$15 per month. Second, in our single plan model, we shall restrict the maximum investment in Flex to \$1250. Recall the "caps" for the Base and Buy-up plans with single coverage are \$1500 and \$1250, respectively. Obviously, any meaningful comparison in our model must limit the Flex investment, denoted f, to the minimum of these two caps; however, it should be understood that although $180 \leq f \leq 1250$ for this model, f would likely be only a portion of a larger Flex reserve which would help defray other medical expenses beyond out-of-pocket, after insurance medical expenses. Again, we will begin by deriving the out-of-pocket expense functions while ignoring premium costs at first.

7. Derivation of $S_i(x)$ with the Flex Option

Out-of-pocket expenses differ greatly from the non-flex option. Clearly, the first "expense" incurred is the amount f invested into the flex account. By federal regulations for Flex type accounts, any money left in the account at the end of the "Flex Year" is forfeited and thus, f results in the initial expense. However, because funds are placed in the account pre-tax, any amount placed in Flex reserve remains untaxed regardless of whether or not the employee uses the funds as intended. Accordingly, the tax savings reduces the original flex reserve f by rf, where r is the employee's federal plus state income tax rate. Further, additional out-of-pocket expenses must be accounted for if the employee has remaining medical expenses after the Flex is used up. These remaining expenses are found using the appropriate plans deductible and coinsurance coverage. Complicating this process greatly, we must consider various flex amounts with respect to the bounds derived earlier in the out-of-pocket expense functions (1) and (2).

We shall begin by examining the out-of-pocket expenses for the Buy-up Plan. Recall, the previous derived function (1):

$$S_1(x) = \begin{cases} x & 0 \le x \le 250 \\ .2x + 200 & 250 < x \le 5250 \\ 1250 & x > 5250 \end{cases}.$$

Should the employee invest less than \$250 in Flex, depending on the exact amount, the investment may not cover the entire deductible should enough expenses occur. On the other hand, if the employee's $f > \$250$, the Flex account will certainly cover the deductible, but may or may not run out before the cap is met. First, assume the employee chooses to invest between the minimum required and the deductible. Thus, $180 \le f \le 250$. Then, when $0 \le x \le f$, because the expenses fall under the deductible, the employee must pay 100% of them, but the flex account will be used to pay all of these expenses. Therefore, the employee incurs out-of-pocket expenses of $f - rf$. If the expenses are greater than amount in Flex, yet still under the deductible such that $f < x \le 250$, then the employee uses the entire amount in Flex, receives the tax savings, and then must pay any additional expenses above the Flex amount, or

$$f - rf + (x - f) = x - rf.$$

If the expenses are such that $250 < x \le 5250$, then as before, the employee incurs the initial expense of the Flex investment, receives the tax savings, must pay 100% of the remaining deductible above the Flex amount, and pay the remaining expenses after coinsurance is considered, or

$$f - rf + (250 - f) + .2(x - 250) = .2x + 200 - rf.$$

Finally, if the employee's medical expenses are greater than \$5250, the out-of-pocket expenses are similar to the those just shown, except the cap is in place. This is shown by

$$f - rf + (250 - f) + .2(5250 - 250) = 1250 - rf.$$

Summarizing, and denoting this function as $S_3(x)$:

$$S_3(x) = \begin{cases} f - rf & 0 \le x \le f \\ x - rf & f < x \le 250 \\ .2x + 200 - rf & 250 < x \le 5250 \\ 1250 - rf & x > 5250 \end{cases},$$

for $180 \le f \le 250$.

Now consider a Flex amount f greater than \$250, but recall our model caps the flex investment at \$1250. In this case, when Flex runs out depends upon the amount put into the account. To find this amount, we must first solve the coinsurance piece of (1) for x when it set equal to f. Thus, $.2x + 200 = f$ implies the Flex account will run out when estimated expenses $x = 5f - 1000$. If $f > 250$, then $5f - 1000 > 250$, so when the expenses are such that $0 \leq x \leq 5f - 1000$, the Flex account will cover the entire \$250 deductible as well as any coinsurance costs until the account runs out. So the employee's out-of-pocket expense is the Flex amount minus the tax savings or $f - rf$. If the expenses are greater than the amount in flex can cover such that $5f - 1000 < x \leq 5250$, the employee incurs the initial cost, receives the tax savings, and then must pay the coinsurance on the remaining expenses, or $f - rf + .2[x - (5f - 1000)] = .2x + 200 - rf$. Finally, if the expenses are greater than \$5250, as before, the expenses are similar to the line above except the cap kicks in. So $f - rf + .2[5250 - (5f - 1000)] = 1250 - rf$. Denoting this function as $S_4(x)$, and restating:

$$S_4(x) = \begin{cases} f - rf & 0 \leq x \leq 5f - 1000 \\ .2x + 200 - rf & 5f - 1000 < x \leq 5250 \\ 1250 - rf & x > 5250 \end{cases},$$

for $250 < f \leq 1250$. Similarly, the Base Plan functions are derived and listed below:

$$S_5(x) = \begin{cases} f - rf & 0 \leq x \leq f \\ x - rf & f < x \leq 500 \\ .5x + 250 - rf & 500 < x \leq 2500 \\ 1500 - rf & x > 2500 \end{cases}$$

for $180 \leq f \leq 500$, and

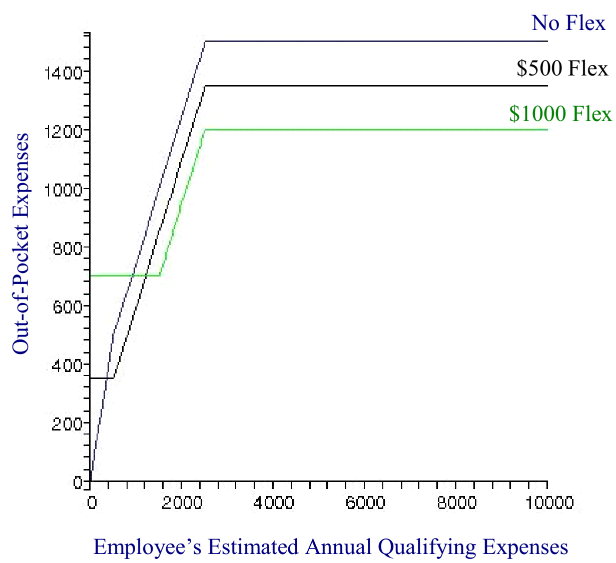
$$S_6(x) = \begin{cases} f - rf & 0 \leq x \leq 2f - 500 \\ .5x + 250 - rf & 2f - 500 < x \leq 2500 \\ 1500 - rf & x > 2500 \end{cases}$$

for $500 < f \leq 1250$. Note that the bound $2f - 500$ is found as before by setting the coinsurance $.5x + 250$ from (2) equal to f and solving for x.

Figure 4 below shows a comparison of the Base Plan expenses with and without Flex. The graph compares the original expense function previously shown in Table 1 with the Base with Flex Plan functions (3) and (4) above. Flex amounts of \$500 and \$1000 are chosen arbitrarily as is a tax rate of 30% (the approximate federal plus state income taxes). The reader should note how as the flex investment increases, the initial out-of-pocket expense to the employee increases. Clearly, investing in Flex reduces the actual expenses paid due to the tax benefits of the program.

Allowing people to pay for medical expenses with money which instead would have been paid as taxes has a significant impact on health insurance costs. An important consideration, however, is for the employee to invest an appropriate amount into the Flex account as to minimize forfeiture. As Figure 3 does not include premium costs, we must now continue the examination of the Flex option by repeating previous methods and develop a cost-comparison function.

Figure 4: Base Plan Expenses Compared



8. Development of Cost-Comparison Function $C_i(x)$ with Flex Considered

Following previous methods, we shall find the difference function to compare the costs of the Base and Buy-up Plans while taking advantage of the Flex option. Unlike before, the Flex account causes additional break points within the piecewise function. We must again examine the previous cost-comparison equation and its bounds and in addition, carefully determine the bounds of f using the expense functions found in the last section. Combining bounds of f, we have:

$$\begin{aligned} 180 &\leq f \leq 250 \\ 250 &< f \leq 500 \\ 500 &< f \leq 1250 \end{aligned}$$

The first set of bounds offers no problems, and thus the Buy-up and Base functions can quickly be subtracted, giving:

$$C_3(x) = \begin{cases} A & 0 \le x \le f \\ A & f < x \le 250 \\ A - .8x + 200 & 250 < x \le 500 \\ A - .3x - 50 & 500 < x \le 2500 \\ A + .2x - 1300 & 2500 < x \le 5250 \\ A - 250 & x > 5250 \end{cases},$$

for $180 \le f \le 250$.

Recall that A is the difference in annual premium costs and note there is no tax savings on any money spent on the premiums. (The first two pieces of this function may be combined, but have been left separate for the reader).

The second set of bounds for the Flex amount is more of a challenge. Recall the amount invested in Flex is completely used when qualified expenses equal $5f - 1000$. Also, if $250 \le f \le 500$, then $250 \le 5f - 1000 \le 1500$, but notice (see $C_3(x)$) the previously determined bound of 500 also falls between 250 and 1500 so we must determine when $5f - 1000$ equals the 500 bound. Setting $5f - 1000 = 500$ implies $f = 300$. We must then separate the second set of bounds for the Flex amount with another break point at 300. Finding the next two cost-comparison functions then results in:

$$C_4(x) = \begin{cases} A & 0 \le x \le f \\ A + f - x & f < x \le 5f - 1000 \\ A - .8x + 200 & 5f - 1000 < x \le 500 \\ A - .3x - 50 & 500 < x \le 2500 \\ A + .2x - 1300 & 2500 < x \le 5250 \\ A - 250 & x > 5250 \end{cases},$$

for $250 < f \le 300$, and

$$C_5(x) = \begin{cases} A & 0 \le x \le f \\ A + f - x & f < x \le 250 \\ A - .5x - 250 + f & 500 < x \le 5f - 1000 \\ A - .3x - 50 & 5f - 1000 < x \le 2500 \\ A + .2x - 1300 & 2500 < x \le 5250 \\ A - 250 & x > 5250 \end{cases},$$

for $300 < f \le 500$.

A similar problem arises in the last set of Flex amount bounds, where $500 < f \le 1250$. Upon carefully subtracting the expense functions, we see once again the $5f - 1000$ bound can be greater than or less than the previously determined bound of 2500. As before, setting $5f - 1000$ equal to 2500 determines exactly when this change occurs. Solving yields $f = 700$. Thus, another break point must be added in the last set of bounds for Flex. Then, subtracting as before, we find the last cost-comparison equations:

$$C_6(x) = \begin{cases} A & 0 \le x \le 2f - 500 \\ A - .5x - 250 + f & 2f - 500 < x \le 5f - 1000 \\ A - .3x - 50 & 5f - 1000 < x \le 2500 \\ A + .2x - 1300 & 2500 < x \le 5250 \\ A - 250 & x > 5250 \end{cases},$$

for $500 < f \le 700$, and

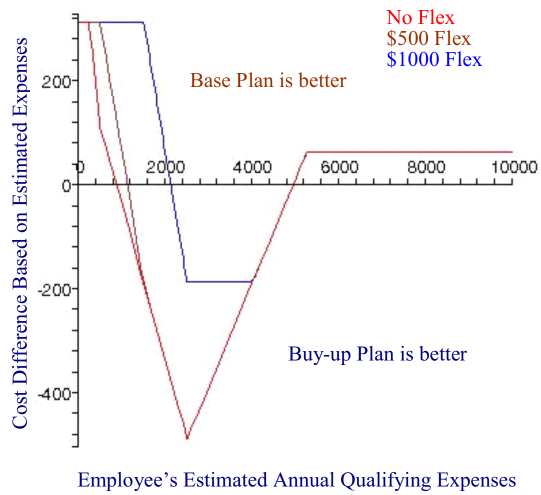
$$C_7(x) = \begin{cases} A & 0 \le x \le 2f - 500 \\ A - .5x - 250 + f & 2f - 500 < x \le 2500 \\ A + f - 1500 & 2500 < x \le 5f - 1000 \\ A + .2x - 1300 & 5f - 1000 < x \le 5250 \\ A - 250 & x > 5250 \end{cases},$$

for $700 < f \le 1250$.

Similar to the previous graphic, Figure 5 offers a comparison between the cost-comparison functions without Flex, and with the same two arbitrarily chosen Flex amounts, \$500 and \$1000 as well as the same tax rate of 30%. Perhaps the greatest feature of this graph is it neatly shows what a judicious use of the Flex account does for the employee. As the Flex amount increases, the range of values for which the Base Plan is more economical also increases. Since the extra premium required to enroll in the Buy-up Plan does not share in the Flex tax savings, this general result is not entirely unexpected. These results, however, can easily be used to determine precisely where the new break even points lie when the Flex modification is included in the model.

Similar methods are used to derive the non-single cost-comparison functions with the added consideration for the Flex account.

Figure 5: Cost-Comparison Flex vs. No Flex



9. Conclusions

The results from this analysis were submitted to Washburn University's Human Resources Department and its Benefits Committee and immediately received great interest from both. In response to a request from the Human Resources Director, the author of this paper developed a web site [2] for the faculty and staff of Washburn to help them make an informed decision about their coverage selection. This site may be found at: The site only considers the non-Flex cases. In addition, the author presented the same findings as well as the results from the Flex modification to a group of faculty as they prepared to make their coverage selection for the coming insurance year.

From these audiences, clearly the results from this analysis, and generally any cost-comparison analysis of insurance plans, are of great benefit to the users. Certainly, one cannot expect most people to perform such an examination of the Flex option, yet the analysis on the basic components of insurance plans should be completed in order to grasp the true out-of-pocket costs in relation to estimated qualified expenses. In fact, the results from analyses like this one should be included as part of all plan coverage and option summaries. Making an informed decision about one's health care benefits everyone.

Acknowledgements: I am grateful to Washburn University's Creative and Scholarly Innovation Committee for the grant received in support of this project. I would also like to thank my peer, Jo Marie Rozzelle for her time, effort, and commitment to this project. In addition, this analysis was supervised by Dr. Al Riveland, whose suggestions, guiding hand, and patience (especially his patience!) were instrumental in its completion. For these and numerous other reasons, he is infinitely appreciated.

Appendix A: Annual Premiums

Note each salary tier pays a lower percentage of the top level for both the Base and Buy-up Plans; however, when the difference in premiums is examined, the opposite is true—the cost to "upgrade" for the better coverage increases as the salary decreases. This is especially true for employees electing Family coverage, where it costs someone at the lowest salary level over twice as much for the Buy-up Plan coverage. This drastic difference leads to the Base Plan always being a more economical option (see Appendix B). One reason given for this is the premiums for the Base Plan do not increase as much as the Buy-up Plan for Family coverage. While the premium percentages remain constant for Buy-up coverage, they actually decrease for Family coverage in the Base Plan thereby causing a significant increase in the difference between the two.

Salary	Single	Employee/ Children	% of Top Tier	Employee/ Spouse	% of Top Tier	Employee/ Family	% of Top Tier
Buy-up Plan (p_1)							
>$48,984	$312.00	$3,315.56	100%	$4,215.36	100%	$8,468.88	100%
$42,328-$48,984	$312.00	$2,984.88	90%	$3,793.80	90%	$7,622.04	90%
$35,360-$48,984	$312.00	$2,653.20	80%	$3,372.24	80%	$6,775.08	80%
$28,600-$35,359	$312.00	$2,321.64	70%	$2,950.80	70%	$5,928.24	70%
$21,944-$28,559	$312.00	$1,989.96	60%	$2,529.24	60%	$5,081.28	60%
<$21,944	$312.00	$1,658.28	50%	$2,107.68	50%	$4,234.44	50%
Base Plan (p_2)							
>$48,984	$0.00	$2,764.56	100%	$3,550.92	100%	$7,572.96	100%
$42,328-$48,984	$0.00	$2,425.56	88%	$3,118.80	88%	$6,535.56	86%
$35,360-$48,984	$0.00	$2,086.68	75%	$2,686.80	75%	$5,498.04	73%
$28,600-$35,359	$0.00	$1,747.68	63%	$2,254.68	63%	$4,460.64	59%
$21,944-$28,559	$0.00	$1,408.80	51%	$1,822.68	51%	$3,423.24	45%
<$21,944	$0.00	$1,069.80	39%	$1,390.56	39%	$2,385.72	32%
Difference $(A = p_1 - p_2)$							
>$48,984	$312.00	$552.00	100%	$664.44	100%	$895.92	100%
$42,328-$48,984	$312.00	$559.32	101%	$675.00	102%	$1,086.48	121%
$35,360-$48,984	$312.00	$566.52	103%	$685.44	103%	$1,277.04	143%
$28,600-$35,359	$312.00	$573.96	104%	$696.12	105%	$1,467.60	164%
$21,944-$28,559	$312.00	$581.16	105%	$706.56	106%	$1,658.04	185%
<$21,944	$312.00	$588.48	107%	$717.12	108%	$1,848.72	206%

Appendix B: Break Even Points

Coverage	Salary Tier	Lower Point	Upper Point
Single	N/A	\$873	\$4940
Employee + Child/ren	$> \$48,984$	\$1507	\$10,240
	$\leq \$48,984$	\$1531	\$10,203
	$\leq \$42,328$	\$1555	\$10,167
	$\leq \$35,360$	\$1580	\$10,130
	$\leq \$28,600$	\$1604	\$10,094
	$\leq \$21,944$	\$1628	\$10,058
Employee + Spouse	$> \$48,984$	\$1881	\$9678
	$\leq \$48,984$	\$1917	\$9625
	$\leq \$42,328$	\$1951	\$9573
	$\leq \$35,360$	\$1987	\$9519
	$\leq \$28,600$	\$2022	\$9467
	$\leq \$21,944$	\$2057	\$9414
Employee + Family	$> \$48,984$	\$2653	\$8520
	$\leq \$48,984$	\$3288	\$7568
	$\leq \$42,328$	\$3923	\$6615
	$\leq \$35,360$	\$4559	\$5662
	$\leq \$28,600$	None	None
	$\leq \$21,944$	None	None

Note that the Base Plan is a more economical option when the employee's estimated expenses are less than the lower bound or greater than the upper bound. The Buy-up Plan is more economical when an employee's expenses are between the bounds. In the case of the two lowest salary levels of the Employee + Family coverage, the Base Plan is always the most economical option.

References

[1] Blue Cross/Blue Shield. Washburn University Employee Health Insurance Coverage Brochure (2005).

[2] `http://www.washburn.edu/admin/human-resources/2004insurance/index.html`

A Mathematical Model for Fibroblast Growth Factor Competition Based on Enzyme Kinetics

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1. Introduction

Fibroblast growth factors (FGFs), among the earliest growth factors to be identified and purified, constitute a large family of at least 25 unique but related secreted proteins that stimulate cell proliferation and that are expressed in many tissues. The levels of FGFs found in these tissues are regulated by many biological factors, which reflects the involvement of the FGFs in more than one physiological event. One important physiological function of FGFs is in wound healing. The role of FGFs in wound healing has been demonstrated in many ways, among them FGFs are found in wound fluids, the absence of FGF-2 delays wound healing [6], and the expression of many FGF genes increases after wounding [8]. In combination with other growth factors, FGFs also play many roles in early embryonic development including to define the dorso-ventral pattern of the neural tube [15], to promote limb development [11], and to define the structure of the early embryo [3]. The FGFs act through specific receptors (FGFR) that initiate signals inside the cell to alter cellular functions such as gene expression. The importance of these receptors to normal development is demonstrated by the many human skeletal diseases caused by mutations in FGFR genes [16].

Four related FGFR genes are the source of 12 different FGF receptor proteins. Each receptor protein binds more than one FGF type, each with a specific affinity that is determined by the receptor-FGF pair. Many studies have shown that more than one FGF is produced in a tissue at the same time. For example, in the skin the genes encoding sixteen unique FGFs are simultaneously active during wound healing. Thus, in vivo, FGF receptors are exposed to more than one FGF at simultaneously. In most cases the cellular response is determined by the nature of the receptor and not by the ligand (FGF), although there are some possible exceptions [5]. However, the response of the receptor depends on the interplay of FGFs present in the environment and their affinities for the receptor. Here we examine a simple case of two FGFs (FGF-1 and FGF-2) interacting with the receptors on a single cell type in cultured cells. Using the biological data, we develop a mathematical model that simulates the competition between these growth factors for the same cell surface receptors. The construct of the pathway of this model looks similar to that of [2] in that the basis for the model is a system of coupled differential equations; however, the underlying mechanism being modeled is different. In [2], the authors examine the effects of FGF-2 and an inhibitor of growth of both primary and secondary tumors; whereas, this study aims to model how the interaction of two fibroblast growth factors affects cell proliferation. After deriving the model, we use simulations in MATLAB and optimization to extrapolate the values of a variety of biochemical parameters imbedded within the model. Finally, we examine use of the model as the basis for a testable hypothesis. We explore this predictive ability with further simulations in MATLAB.

2. Biological Activity of FGF-1 and FGF-2

In [9], Neufeld and Gospodarowicz examined the physical and chemical characteristics of FGF-1 and FGF-2.[7] Noting the apparent similarities between FGF-1 and FGF-2, Neufeld and Gospodarowicz proceeded to investigate the differential affinities of these two FGFs to the same cell surface receptor. Several experiments were carried out to characterize biological activity of FGF-1 and FGF-2. Specifically, the effects of increasing concentrations of either FGF-1 and FGF-2 on cell proliferation were observed. Neufeld and Gospodarowicz began with plates each containing 4 x 10^4 cells from a baby hamster kidney cell line (BHK-21). One set of plates was exposed to increasing concentrations of FGF-1 (ranging from 50 pg/mL to 250 ng/mL), while another set of plates was exposed to increasing concentrations of FGF-2 (ranging from 2.5 pg/mL to 25 ng/mL).

[7] At the time [9] was in publication, FGF-1 and FGF-2 were referred to as acidic and basic fibroblast growth factor, respectively. For comprehensibility we will continue to employ the numerical notation to refer to these FGFs.

These increments of growth factor were added in two boluses, one on day 0 and one on day 2. After 4 days, the number of cells on each plate was counted and recorded. These data were displayed as FIG 2 in [9], recreated here as Figure 1.

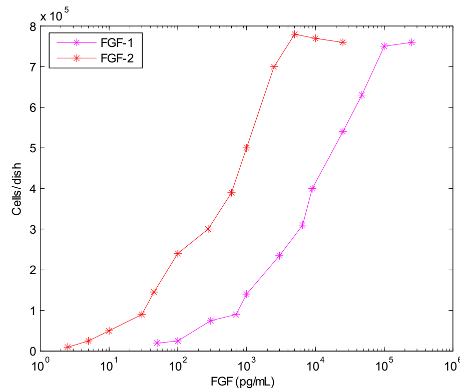

Figure 1. Effects of FGF-1 and FGF-2 Concentration on Proliferation of BHK-21 Cells (from [9])

3. Biochemical Kinetics

The first competitive pathway can be described as follows: Suppose R is a free receptor on a BHK-21 cell capable of being activated by either FGF-1 or FGF-2. Let G^1 be a molecule of FGF-1. Then, the binding of FGF-1 to a free receptor leads to an intermediate complex, $\{RG^1\}$, which releases a product, call it P^1, by the mechanism:

$$\begin{aligned} R + G^1 &\underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \{RG^1\}, \\ \{RG^1\} &\overset{k_2}{\rightarrow} P^1. \end{aligned} \tag{1}$$

The product P^1 begins a tyrosine-kinase signal transduction pathway leading to an increase in cell number.[8] Concurrently occurring is the binding of FGF-2, G^2, to another free receptor, R. This binding also leads to an

[8] The exact pathway leading to increased cell proliferation is long and involved. For the present discussion, it suffices that the intermediate complex begins a signal transduction pathway ultimately resulting in increased proliferation; hence, this simplification is used for the present model.

intermediate complex, $\{RG^2\}$, which again releases a product, P^2:

$$\begin{aligned} R + G^2 &\underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} \{RG^2\}, \\ \{RG^2\} &\overset{k_4}{\to} P^2. \end{aligned} \tag{2}$$

Product P^2 also initiates a signal transduction pathway. This cascade again results in increased cell number. When both species of growth factor are present, the interplay of these two equations, (1) and (2), results in competition of both species for the same free receptors:

$$\begin{array}{lllll} R & + G^1 \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \{RG^1\} \overset{k_2}{\to} P^1 \\ + & \\ G^2 & \\ k_3 \downharpoonleft\upharpoonright k_{-3} & \\ \{RG^2\} & \\ \downarrow k_4 & \\ P^2 & \end{array} \tag{3}$$

Table 1 summarizes the species present in this pathway.

Table 1. **Notation for Species in Kinetic Equations**

Species	**Notation**
free receptor	R
fibroblast growth factor, FGF-1	G^1
fibroblast growth factor, FGF-2	G^2
product initiating cell proliferation	P^1
product initiating cell proliferation	P^2

This competitive pathway is further developed by writing down the laws of mass action[9] for (1) and (2), as follows:[10]

$$\begin{aligned}\frac{d[G^1]}{dt} &= k_{-1}[\{RG^1\}] - k_1[R][G^1],\\ \frac{d[\{RG^1\}]}{dt} &= -(k_{-1}+k_2)[\{RG^1\}] + k_1[R][G^1],\\ \frac{d[G^2]}{dt} &= k_{-3}[\{RG^2\}] - k_3[R][G^2],\\ \frac{d[\{RG^2\}]}{dt} &= -(k_{-3}+k_4)[\{RG^2\}] + k_3[R][G^2].\end{aligned} \tag{4}$$

At this point we employ the Michaelis-Menten steady state assumption explained in [14]. Essentially, this assumption states that following the initial stage of the reaction, termed the transient phase, the rate of synthesis of an intermediate remains approximately equal to the rate of consumption of said intermediate until the substrate, or growth factor in the present example, is nearly exhausted. Thus, a quasi-equilibrium is reached. Applying this hypothesis, we take the concentrations of both intermediates to be constant and using the notation $K_M^i = (k_{2i} + k_{-(2i-1)})/k_{2i-1}$ for each Michaelis constant, the second and fourth equations in (4) become:

$$\begin{aligned}[\{RG^1\}] &= \frac{[R][G^1]}{K_M^1},\\ [\{RG^2\}] &= \frac{[R][G^2]}{K_M^2}.\end{aligned} \tag{5}$$

9 In this paper, we employ the chemical convention whereby [A] denotes the local concentration of species A in micromoles per liter, or micromolarity.

10 It is important to remark that the first and third equations in (4) have been simplified. Taking into account cell expression and FGF turnover rate, these equations are more completely written as:

$$\begin{aligned}\frac{d[G^1]}{dt} &= k_{-1}[\{RG^1\}] - k_1[R][G^1] + \sigma_{G^1}[R]_T - \mu_{G^1}[G^1],\\ \frac{d[G^2]}{dt} &= k_{-3}[\{RG^2\}] - k_3[R][G^2] + \sigma_{G^2}[R]_T - \mu_{G^2}[G^2],\end{aligned}$$

where σ_{G^1} and σ_{G^2} are constants for cellular expression of G^1 and G^2, respectively, and μ_{G^1} and μ_{G^2} are decay rates for the aforementioned growth factors, and $[R]_T$ is the total concentration of receptors. We may neglect σ_{G^1} and σ_{G^2} because the expression of either growth factor by the BHK-21 cells is negligible relative to the concentration of growth factor being added into the cell cultures. Likewise, we may neglect μ_{G^1} and μ_{G^2} because the decay rate of either growth factor is negligible relative to the concentration of growth factor being consumed by the growing cell populations. Essentially, FGF-1 and FGF-2 are being consumed by the cells at a far faster rate than either half-life would allow for decay.

Substituting the first and second equations of (5) into the first and third equations of (4), respectively, yields:

$$\frac{d[G^1]}{dt} = k_{-1}K_M^1[R][G^1] - k_1[R][G^1] = -\frac{k_2}{K_M^1}[R][G^1],$$
$$\frac{d[G^2]}{dt} = k_{-3}K_M^2[R][G^2] - k_3[R][G^2] = -\frac{k_4}{K_M^2}[R][G^2]. \tag{6}$$

Next, we relate cell density to receptor concentration. We assume, as noted in [2], that the number of BHK-21 cells per unit volume is proportional to the total number of receptors that can initiate a signal transduction pathway in response to a growth factor. Thus, we may write:

$$[N] = \kappa[R]_T, \tag{7}$$

where [N] denotes the concentration of BHK-21 cells, $[R]_T$ denotes the total concentration of receptors, and κ is the proportionality constant. Substituting R_0/N_0 for the proportionality constant κ, we may write:

$$[R]_T = R_0\frac{[N]}{N_0}, \tag{8}$$

where N_0 is the carrying capacity of the BHK-21 cells and R_0 is the total number of receptors at carrying capacity. As [2] explains, we may take R_0 to be on the order of unity, thus our relationship becomes:

$$[R]_T = \frac{[N]}{N_0}. \tag{9}$$

Furthermore, we may write the total concentration of receptors as follows:

$$[R]_T = [R] + [\{RG^1\}] + [\{RG^2\}]. \tag{10}$$

Substituting the first and second equations of (5) into (10) yields:

$$[R]_T = [R] + \frac{[R][G^1]}{K_M^1} + \frac{[R][G^2]}{K_M^2}. \tag{11}$$

Solving for free receptors, $[R]$, gives:

$$[R] = \frac{[R]_T}{1 + \frac{[G^1]}{K_M^1} + \frac{[G^2]}{K_M^2}}. \tag{12}$$

Substitution of (8) into (12) yields:

$$[R] = \frac{\dfrac{[N]}{N_0}}{1 + \dfrac{[G^1]}{K_M^1} + \dfrac{[G^2]}{K_M^2}}. \tag{13}$$

Finally, (13) can be substituted into the first and second equations of (6), as follows:

$$\begin{aligned} \frac{d[G^1]}{dt} &= \left(\frac{-k_2 \dfrac{[G^1]}{K_M^1}}{1 + \dfrac{[G^1]}{K_M^1} + \dfrac{[G^2]}{K_M^2}} \right) \frac{[N]}{N_0}, \\ \frac{d[G^2]}{dt} &= \left(\frac{-k_4 \dfrac{[G^2]}{K_M^2}}{1 + \dfrac{[G^1]}{K_M^1} + \dfrac{[G^2]}{K_M^2}} \right) \frac{[N]}{N_0}. \end{aligned} \tag{14}$$

Describing cell proliferation is slightly more complex but accomplished when several biological considerations are taken into account. First, we assume that cell proliferation is logistic as determined from the characteristic shape of Figure 1. Secondly, as noted in [2], it is reasonable to assume that BHK-21 cell mitosis depends on the concentrations of both growth factors and BHK-21 cell apoptosis is linear in cell density. These considerations allow us to write:

$$\frac{d[N]}{dt} = \phi(G^1, G^2)[N] \left(1 - \frac{[N]}{N_0}\right) - \mu[N], \tag{15}$$

where $\phi(G^1, G^2)$ is the coefficient of the logistic term and μ is the decay rate of BHK-21 cells. The term $\phi(G^1, G^2)$ is a measure of how the growth factors influence mitosis. In the present model, $\phi(G^1, G^2)$ takes the form:

$$\phi(G^1, G^2) = \lambda \left(\frac{\dfrac{[G^1]}{K_M^1} + \dfrac{[G^2]}{K_M^2}}{1 + \dfrac{[G^1]}{K_M^1} + \dfrac{[G^2]}{K_M^2}} \right). \tag{16}$$

As explained in [2], the underlying idea is that sufficient concentrations of either growth factor are necessary for the birth rate to exceed the death rate, but the effects of FGF-1 and FGF-2 on birth rate at saturation of either growth factor are limited to a maximum value of λ. Thus, the equation for cell proliferation becomes:

$$\frac{d[N]}{dt} = \lambda[N]\left(1-\frac{[N]}{N_0}\right)\left(\frac{\frac{[G^1]}{K_M^1}+\frac{[G^2]}{K_M^2}}{1+\frac{[G^1]}{K_M^1}+\frac{[G^2]}{K_M^2}}\right) - \mu[N]. \qquad (17)$$

Combining this equation with the equations in (14), we obtain a predictive model described by a system of three coupled differential equations:

$$\begin{aligned}
\frac{d\,[N]}{dt} &= \lambda\,[N]\left(1-\frac{[N]}{N_0}\right)\left(\frac{\frac{[G^1]}{K_M^1}+\frac{[G^2]}{K_M^2}}{1+\frac{[G^1]}{K_M^1}+\frac{[G^2]}{K_M^2}}\right) - \mu\,[N]\,,\\
\frac{d\,[G^1]}{dt} &= \left(\frac{-k_2\frac{[G^1]}{K_M^1}}{1+\frac{[G^1]}{K_M^1}+\frac{[G^2]}{K_M^2}}\right)\frac{[N]}{N_0}, \qquad (18)\\
\frac{d\,[G^2]}{dt} &= \left(\frac{-k_4\frac{[G^2]}{K_M^2}}{1+\frac{[G^1]}{K_M^1}+\frac{[G^2]}{K_M^2}}\right)\frac{[N]}{N_0}.
\end{aligned}$$

4. Simulations and Optimization

Now that we have constructed a model for the competitive pathway described in (3), we use MATLAB to simulate the experiments performed by Neufeld and Gospodarowicz in [9].

We use the MATLAB solver ODE15s for simulations. First, we simulate the initial trial performed by Neufeld and Gospodarowicz in [9]. In this trial, cell plates containing $4\text{x }10^4$BHK-21 cells were exposed to increasing concentrations of FGF-1 while no FGF-2 was present. The added amounts of FGF-1 are shown in the first column of Table 2.

Thus, the initial conditions for our model are $[N] = 4\text{x }10^4$, $[G^1] =$column 1 of Table 2, and $[G^2] = 0$. Furthermore, $[G^2] = 0$ for the equations in (18) because no FGF-2 is present. This observation means that in this particular trial, the model does not depend on the values of $k_4$and $K_M^2$from the second equation of (18). However, the model does require values for the parameters μ, λ, N_0, k_2, and K_M^1. These values were approximated in [2] and are shown in Table 3.

Table 2. **Concentrations of FGF-1 and FGF-2 for Experiment 1 -** Added Day 0 and Day 2

FGF-1 concentration (no FGF-2 present)	**FGF-2 concentration** (no FGF-1 present)
50 pg/mL	2.5 pg/mL
100 pg/mL	5 pg/mL
300 pg/mL	10 pg/mL
700 pg/mL	30 pg/mL
1 ng/mL	45 pg/mL
3 ng/mL	100 pg/mL
6.5 ng/mL	275 pg/mL
9 ng/mL	600 pg/mL
25 ng/mL	1 ng/mL
47.5 ng/mL	2.5 ng/mL
100 ng/mL	5 ng/mL
250 ng/mL	10 ng/mL
	25 ng/mL

Table 3. **Numerical Values of Parameters Used in Simulations**

Parameter	**Numerical Value (from [2])**
μ	1.0 x 10^{-2} h^{-1}
λ	6.4 x 10^{-1} h^{-1}
N_0	775,000 cells
k_2	1.7 h^{-1}
K_M^1	1.83 x 10^{-2} μM
k_4	1 x 10^{-1} h^{-1}
K_M^2	1.19 x 10^{-2} μM

We now use the ODE15s solver to find the concentration of FGF-1 at time 48 hours.[11] To this concentration of FGF-1 we add the second bolus of growth factor, again expressed in column 1 of Table 2. Finally, we use the solver to determine the number of BHK-21 cells at time 72 hours.

Using a similar method, we simulate increasing concentrations of FGF-2. In this trial there is no FGF-1 present, or $[G^1] = 0$, and the model is not dependent upon the values of $k_2$and $K_M^1$from the equations in (18). Instead, this model utilizes the parameters μ, λ, N_0, k_4, and K_M^2. Numerical values for these parameters were again supplied by [2] and are given in Table 3. Again, by solving the system of differential equations twice, employing the pulse of additional growth factor described earlier, we ob-

[11] This simulation uses hours for the time scale, as opposed to days in [9].

tain an approximation of the biological data. The data for both trials are plotted along with the associated biological data from Figure 1 in Figure 2.

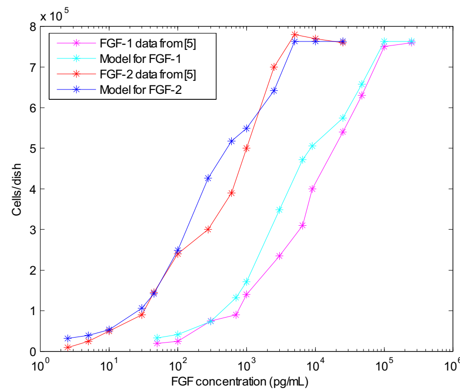

Figure 2. Initial Fit of Model to Biological Data from [9]

We now employ optimization to extrapolate the numerical values of the parameters appearing in the model. In the first trial of this experiment these parameters are μ, λ, N_0, k_2, and K_M^1. In order to find the values of these parameters which give the closest fit to the actual biological data, we first define an error function. This function is the sum of the squares of the differences of the biological data for cell density and the data calculated from the model for cell density, as represented below:

$$E = \sum_{i=1}^{n} (N_i^{\text{exp}} - N_i^{\text{model}})^2. \tag{19}$$

This error function has the values of the parameters as inputs. Different values for the parameters yield a different numerical value for the error function. Then, using a tool in MATLAB known as fminsearch, we minimize the error function and the resultant output is a vector of the values of the parameters which give the closest fit to actual biological data. Using fminsearch for the first trial, the resulting coefficient vector is:

$$\begin{bmatrix} 0.014162 & 0.468937 & 790,568 & 1.952589 & 0.012074 \\ \end{bmatrix},$$

which corresponds to the values for μ, λ, N_0, k_2, and K_M^1. Likewise, we apply an error function to the second trial. Here, we are searching for the values of the parameters μ, λ, N_0, k_4, and K_M^2:

$$\begin{bmatrix} 0.019587 & 0.775144 & 789,977 & 0.06669 & 0.011126 \end{bmatrix}.$$

The revised model, taking into account the optimal values of the parameters, is plotted along with the accompanying biological data in Figure 3.

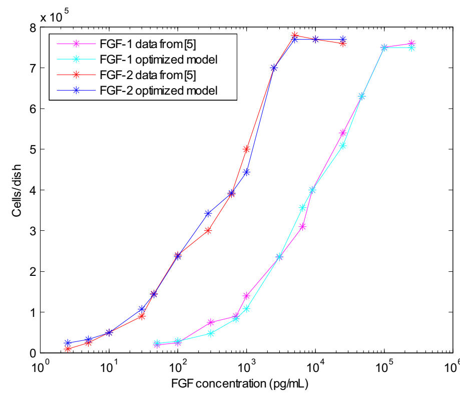

Figure 3. Optimization Fit of Model to Biological Data from [9]

Now we must consider the overlap between the two trials. The individual optimizations yielded slightly different values of μ, λ, and N_0, as shown in Table 4.

Table 4. **Comparison of Shared Parameters**

Parameter	**FGF-1 trial**	**FGF-2 trial**
μ	0.014162 h^{-1}	0.019587 h^{-1}
λ	o.468937 h^{-1}	0.775144 h^{-1}
N_0	790,568 cells	789,977 cells

Using a combined error function where the parameters are defined only once should give a compromise fit for the two trials. This combined error function yields the coefficient vector:

$$\begin{bmatrix} 0.020241 & 0.64678 & 810,667 & 1.3847 & 0.02925 & 0.077062 & 0.012556 \end{bmatrix},$$

which corresponds to parameters $\mu, \lambda, N_0, k_2, K_M^1, k_4$, and K_M^2. Figure 4 shows a plot of the model utilizing these parameters.

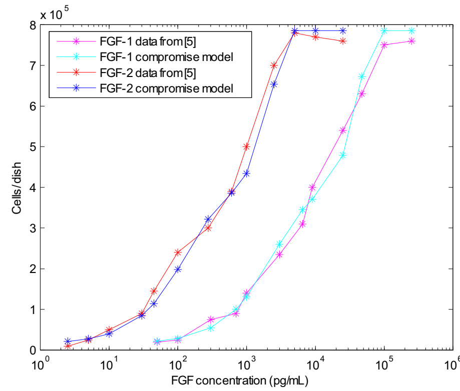

Figure 4. Compromise Fit of Optimization Model to Biological Data from [9]

5. Discussion and Future Work

A number of findings can be drawn from our model. First, our model gives a new perspective on the role of K_M^i. Our model demonstrates that k_2 and k_4 are the driving force and not K_M^1 and K_M^2 as previously thought. This result has a very important implication. It shows that k_2 and k_4 are not always insignificant and this fact must be taken into consideration before simply disregarding the values of these parameters. Moreover, the fact that k_2 and k_4 are significant greatly affects the difference between K_M^i and K_D^i, the dissocation constant. This result again has implications for future research.

Finally, our model demonstrates the importance of parameter estimation in the modeling of biological phenomena. Slightly changing the values of parameters embedded in mathematical models can result in noticeable changes in the fit of the model. This result was shown with the optimizations we performed. Now that we have constructed our model and the parameters have been accurately estimated, we can use the model to make predictions. From our model we are able to formulate several testable hypotheses using MATLAB. We hypothesize about the appearance of several variations of the original experiment.

First, we predict the outcome if the number of pulses is changed. Figure 5 shows the results of a replication of the experiment, the only difference being that in the first trial growth factor is added only initially, the second trial is the exact procedures of the experiment, and the third trial is a pulse of growth factor added initially and consecutively each of the next three days. Each of these trials still involved counting cell number at four days and the same total amount of growth factor was added for all three. Only the average pulse size was varied for each trial.

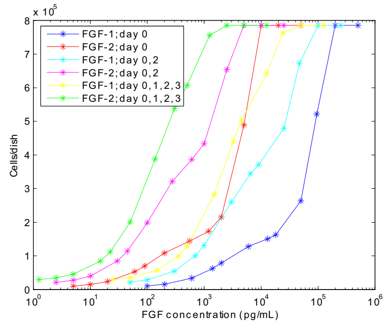

Figure 5. Simulation of Original Experiment Showing Varying Number of Pulses (Counting Day 4)

Next, we predict the outcome of changing the number of days we wait before counting. Figure 6 compares counting day 4 versus day 6 when pulses added initially, day 1, day 2, and day 3.

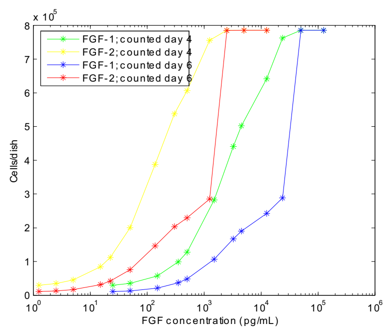

Figure 6. Simulation of Original Experiment Showing Pulses Day 0, 1, 2, and 3 (Counting Either Day 4 or Day 6)

Here we notice that the greatest cell density occurs when counting earlier (day 4) rather than waiting to count (day 6). This could be attributable to either decay of the growth factor or of the BHK-21 cells.

Next, we compare the effects of adding growth factor on consecutive or alternating days and then counting on day 6, as shown in Figure 7.

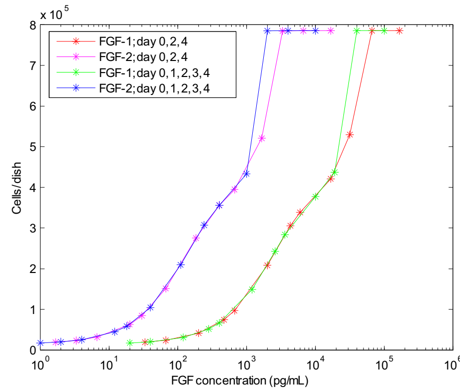

Figure 7. Simulation of Original Experiment Showing Consecutive vs. Alternating Pulses (Counting Day 6)

Here we observe that the model predicts that the trials will initially overlap and then diverge later in the experiment. This could be attributable to the growth factor decaying in the alternating trial while the consecutive trial maintains enough growth factor to last for a longer duration.

Finally, we predict the effects of adding growth factor for an increasing number of pulses. As Figure 8 shows, successive trials attain greater cell density when growth factor is added for a greater number of days.

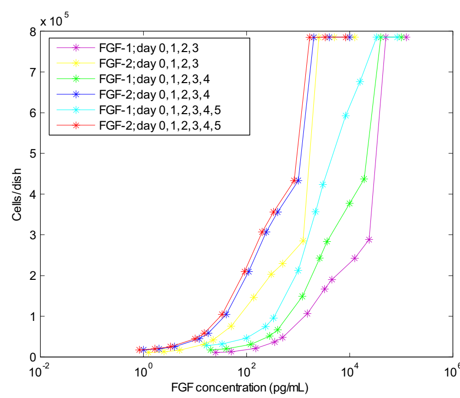

Figure 8. Simulation of Original Experiment Showing Increasing Number of Pulses (Counting Day 6)

Here we have demonstrated that the interaction of multiple growth factors with cell surface receptors can be modeled to produce predictable outcomes. Our model correctly describes the results of experiments performed in [9] and can predict the outcome of many experimental protocols, given accurate parameters for modeling. Although the current model was developed to simulate a relatively simple cell culture system with only two growth factors and one receptor, its capacity for expansion to include more growth factors and growth factor receptors identifies this model as an excellent base for developing testable simulations of complex biological systems. The development of predictive models is essential to understanding the complex interplay of growth factors and their receptors, as happens during embryonic development and wound healing.

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An Algorithm for Evaluating Farkel Strategies

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1. Introduction

The dice game farkel is an extremely simple game to learn how to play. However, like most good games, farkel requires a balance of luck and strategy to win. In this paper, I will outline a method of evaluating those strategies. This evaluation will be done by finding the long term average number of points that each strategy produces.

2. Rules of Farkel

In order to discuss the strategies of farkel, one must first have some background on how farkel is played.

Before the game begins, each player is given a list of possible "hands." Examples of these hands are four-of-a-kind's, straights, and a pair of three-of-a-kind's. The most important two of these hands is a single "1" or a single "5." A "1" by itself is worth 100 points, and a "5" by itself is worth 50. Generally, these are the most frequently seen hands. All the other single dice are worth nothing by themselves. For an example of a full list of hands and points, see Figure 1.

A Sample Scoring System

- Single 1: 100
- Single 5: 50
- Three-of-a-kind:
  - Three ones: 1000
  - Three twos: 200
  - Three threes: 300
  - Three fours: 400
  - Three fives: 500
  - Three sixes: 600
- Four-of-a-kind: 2x the corresponding value for three-of-a-kind's
- Five-of-a-kind: 4x the corresponding value for three-of-a-kind's
- Six-of-a-kind: 8x the corresponding value for three-of-a-kind's
- Straight (1-2-3-4-5-6): 1500
- Two three-of-a-kind's: 1750
- Three pairs: 1000

Note that a four-of-a-kind and a pair can also be thought of as three pairs. The player should take the interpretation that is worth more points.

Figure 1. This system has been modified so that the points are distributed "fairly." The original form is available at [3].

At the beginning of their turn, a player is given six dice. The player rolls the dice, and earns points according to the list of acceptable hands. If there are any remaining dice after this first hand is rolled, the player can choose to either continue rolling with the dice that are left, or end their turn and pass the dice onto the next player. If they should ever run out of dice, the player then gets to start over with all six dice and continue to build up points. However, if the player should ever roll, and none of the acceptable hands are rolled, then the player has "farkeled." They lose all the points that they have earned on that turn, and play resumes with the next player. Play continues until one player reaches a certain preset value, generally 10,000 points.

Hence, as a player's turn progresses, they gradually have fewer and fewer dice to roll. As a result the probability of farkelling rises dramatically. So at each stage, a player must balance earning more points and losing everything. Further, after each roll of the dice, the player can choose not to keep everything that they have rolled, provided that they keep something. In some instances, not keeping lower valued hands pays off by giving the player more dice to roll.

3. Background Data

In order to compute the average points earned, certain probabilities are required. For example, if one rolls six dice, what is the probability of rolling a three-of-a-kind? A four-of-a-kind? What about possible combinations of hands like a three-of-a-kind and a pair of 1's? The same data needs to be found for rolling five dice, four dice, and so on. Note that all of these are treated as isolated events. Nothing about subsequent roles is considered at this time.

We shall only consider the probability distribution of rolling six dice in this paper. All other cases can be determined with the exact same method presented here.

When rolling six dice, there are $6^6 = 46656$ different hands that can be rolled. This is a daunting number to try to work with directly. Unfortunately, this number also counts many hands multiple times by considering the "order" of the dice rather than just hand that was rolled. That is to say, this number considers rolling a 1-2-3-3-3-3, different than rolling a 2-1-3-3-3-3. Thus, some work will be required to break this into more manageable parts.

To begin, we consider all the different ways to partition 6 with positive integers. Each number of these partitions will correspond to rolling the same value multiple times. For example, we can write $6 = 2 + 2 + 1 + 1$. To this partition we associate rolling a two-of-a-kind, another two-of-a-kind, a one-of-a-kind, and yet another one-of-a-kind. Once we have all of these partitions determined, we can use standard counting techniques to find the number of possible hands without considering the permutations.

With a minimal amount of effort, we get the following list of partitions:

$$\begin{array}{ccc} 6 & 5+1 & 4+2 \\ 4+1+1 & 3+3 & 3+2+1 \\ 3+1+1+1 & 2+2+2 & 2+2+1+1 \\ 2+1+1+1+1 & 1+1+1+1+1+1 & \end{array}$$

Consider once again the partition $2+2+1+1$. There are four values that need to be selected to completely describe each of these hands—one for each of the two-of-a-kind's, and one for each of the one-of-a-kind's. There are $\binom{6}{2}=15$ ways to select what values correspond to the two two-of-a-kind's. There remain four values from which to select the two one-of-a-kind's. Hence there are $\binom{4}{2}=6$ ways to select the one-of-a-kind's. Therefore, there are a total of $\binom{6}{2}\binom{4}{2}=15\cdot 6=90$ hands that correspond to the $2+2+1+1$ partition. Similarly, we can calculate the number of hands associated with each partition as shown in Table 1.

Partitions	Number of hands
6	$\binom{6}{1}=6$
$5+1$	${}_6P_2=30$
$4+2$	${}_6P_2=30$
$4+1+1$	$\binom{6}{1}\binom{5}{2}=60$
$3+3$	$\binom{6}{2}=15$
$3+2+1$	${}_6P_3=120$
$3+1+1+1$	$\binom{6}{1}\binom{5}{3}=60$
$2+2+2$	$\binom{6}{3}=20$
$2+2+1+1$	$\binom{6}{2}\binom{4}{2}=90$
$2+1+1+1+1$	$\binom{6}{1}\binom{5}{4}=30$
$1+1+1+1+1+1$	$\binom{6}{6}=1$

Table 1

Now we have reduced our number of hand from 46656 to 462 different hands! We can do better still.

Once again consider the $2+2+1+1$ partition. First notice that there are $\frac{6!}{2!\cdot 2!\cdot 1!\cdot 1!}=180$ permutations of each of the 90 hands. So the $2+2+1+1$ partition accounts for $\frac{90\cdot 180}{46656}=\frac{25}{72}$ of all possible hands. We can now use the farkel scoring system to our advantage. Since we are only interested in hands that score points, we can ignore the remaining hands. Using - , $*$, $\times$, and $\star$ to represent dice that do not affect the score, we can list out all the hands corresponding to the $2+2+1+1$ partition. These hands are:

1 1 5 5 - $*$	1 1 - - 5 $*$	1 1 - - $* \times$	5 5 - - 1 $*$	5 5 - - $* \times$
- - $* *$ 1 5	- - $* *$ 1 $\times$	- - $* *$ 5 $\times$	- - $* * \times \star$	

Once again using counting techniques, we can determine how many hands out of the possible 90 hands for which each of these accounts. For example, consider the case "1 1 5 5 - $*$ ". Since the "1 1 5 5" portion is fixed, the "- $*$ " is the only part that allows for any variation. Further, there are 4 dice left to fill these two spots, since 2, 3, 4, and 6 are all worth no points. Hence there are $\binom{4}{2} = 6$ possible hands that account for "1 1 5 5 - $*$". Finally, since this is $\frac{6}{90} = \frac{1}{15}$ of the $2+2+1+1$ partition, the probability of rolling this particular hand is $\frac{1}{15} \cdot \frac{25}{72} = \frac{5}{216}$. After repeating this process for all of the $2 + 2 + 1 + 1$ partition, we get Table 2.

Hand	Number Possible	Total Probability
1 1 5 5 - $*$	$\binom{4}{2} = 6$	$\frac{5}{216}$
1 1 - - 5 $*$	${}_4P_2 = 12$	$\frac{5}{108}$
1 1 - - $* \times$	$4 \cdot \binom{3}{2} = 12$	$\frac{5}{108}$
5 5 - - 1 $*$	${}_4P_2 = 12$	$\frac{5}{108}$
5 5 - - $* \times$	$4 \cdot \binom{3}{2} = 12$	$\frac{5}{108}$
- - $* *$ 1 5	$\binom{4}{2} = 6$	$\frac{5}{216}$
- - $* *$ 1 $\times$	$4 \cdot \binom{3}{2} = 12$	$\frac{5}{108}$
- - $* *$ 5 $\times$	$4 \cdot \binom{3}{2} = 12$	$\frac{5}{108}$
- - $* * \times \star$	$\binom{4}{2}\binom{2}{2} = 6$	$\frac{5}{216}$

Table 2

Repeating this process with all of the partitions of 6, we get the complete distribution of rolling 6 dice. This process reduces everything to 131 cases. The results for the remainder of the partitions, along with the results for 5, 4, 3, 2, and 1 dice are available by contacting Dr. Brian Hollenbeck at `bhollenb@emporia.edu`.

Once all these probabilities have been collected, one can assign to each hand its value according to one's particular strategy and scoring system. After each hand is rolled, a certain amount of dice remains to be played. I shall refer to this as the "remainder" of that hand. After each hand's score is computed, its remainder is also determined. For our purposes, we need to have all the hands and associated points sorted by remainder. The result of sorting the data when no particular strategy is also available by contacting Dr. Hollenbeck at the above e-mail address.

We shall need to use these sorted probabilities later. In order to coherently discuss them, we shall need to give symbols to each of them. After one rolls the dice, some, all, or none of the dice are available to rerolled. We shall need to know the probability that a specific remainder is left over. This shall be denoted $P_j^i = P(i \text{ dice remain} \mid j \text{ dice were rolled})$. Further, we shall need to know the average number of points earned depending on how many of the dice remain. This shall be denoted $E_j^i = E(X \mid j \text{ dice rolled}, i \text{ remain})$. Note that neither of these values in any way considers what happens in subsequent rolls.

4. Tools from Probability

As a brief detour before the development of the algorithm, we need some standard tools from basic probability and a few nonstandard tools as well.

Theorem 1 (Law of Total Probability) *Suppose that $A_1, A_2, \ldots, A_n$ is a sequence of disjoint events that also exhausts the sample space of a probability distribution. For any event B,*

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i),$$

where $P(B \mid A)$ is the conditional probability of B given A.

Hence, we can determine the probability of an event by breaking it into smaller events. The proof of Theorem 1 is in many standard probability textbooks. See [1, Theorem 1.5.2], for example. Using the Law of Total Probability as inspiration, we can also derive the following:

Theorem 2 *Let $A_1, A_2, \ldots, A_n$ be a sequence of disjoint events that exhausts the sample space of a discrete probability distribution, and let X be a random variable defined over that distribution. Then*

$$E(X) = \sum_{i=1}^{n} P(A_i)E(X \mid A_i).$$

Proof. Using the definition of $E(X)$ and the Law of Total Probability:

$$E(X) = \sum_{\text{all } x} x \cdot P(X = x) = \sum_{\text{all } x} x \left(\sum_{i=1}^{n} P(A_i)P(X = x \mid A_i) \right)$$

Changing the order of summation gives

$$\begin{aligned} E(X) &= \sum_{\text{all } x} \left(\sum_{i=1}^{n} xP(A_i)P(X = x \mid A_i) \right) \\ &= \sum_{i=1}^{n} \left(\sum_{\text{all } x} xP(A_i)P(X = x \mid A_i) \right) \\ &= \sum_{i=1}^{n} \left(P(A_i) \sum_{\text{all } x} xP(X = x \mid A_i) \right) = \sum_{i=1}^{n} P(A_i)E(X \mid A_i) \end{aligned}$$

The change in the order of summation is clearly justified if the sum over x is finite. However, if the sum over x is infinite, we must rely on a theorem from calculus. Recall that if $\sum_{j=1}^{\infty} x_j$ and $\sum_{j=1}^{\infty} y_j$ both converge, then $\sum_{j=1}^{\infty}(x_j + y_j)$ converges to $\sum_{j=1}^{\infty} x_j + \sum_{j=1}^{\infty} y_j$. By induction, we can extend this to the sum of n infinite series. Hence our change of order is indeed justified. ■

Hence, we can also determine expected value by considering smaller pieces of the overall event. Finally, we need a tool for evaluating the expected value of certain joint probabilities.

Theorem 3 *Suppose $X_1, \ldots, X_n$ are random variables. Then*

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i).$$

Once again, this is a standard theorem and its proof is in many probability texts. For a proof, see [1, Theorem 5.2.2] or [2, Theorem 3 in §5.3].

5. Development of the Algorithm

We ultimately want to get the expected value of one's score. That is, we want to find the long-run average score. To facilitate this, we make a few starting assumptions. We first assume that the player has a fixed stopping point. If there are ever n or fewer dice remaining to be rolled, the player will choose to stop rather than continue rolling. Through the remainder of this paper, n shall refer to this stopping point. Further, we assume that the decision on which dice to keep for each hand is already predetermined, as this was figured into the P_j^i's and E_j^i's.

Evaluating the long-run average score directly would be difficult and time consuming. However, by considering small pieces, this work can be greatly reduced. Let us begin by considering the big picture. Once a player starts to roll the first six dice, there are three events that can occur before that player gets a chance to roll all six again. The player can either farkel, stop rolling, or use all the dice and be ready to roll again. Once they are ready to roll again, the player faces the same three events. Figure 2 gives a flowchart of this. Hence, if we can get a handle on the first three possibilities (see Figure 3), we can use this to determine the overall expected value.

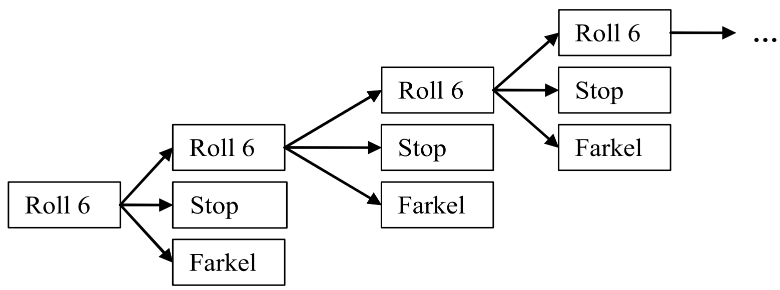

Figure 2

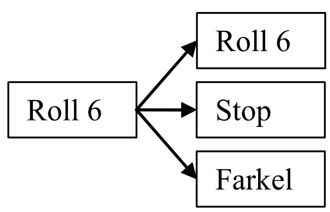

Figure 3

As a first step towards understanding the situation expressed in Figure 3, we shall calculate the expected value of the entire structure. That is, we shall find the average score earned by just rolling 6 dice without considering rerolling the dice. To give an idea of what is going on in this case, let us again diagram what happens. However, for space and simplicity, let us only consider what would happen if we began with three dice. This is shown in Figure 4.

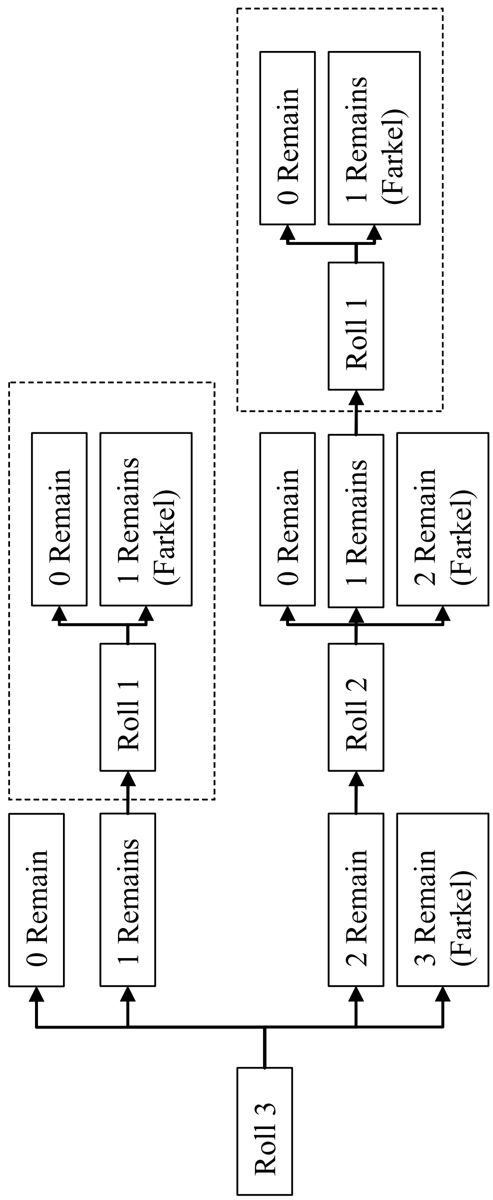
Roll 3
0 Remain
1 Remains
2 Remain
3 Remain (Farkel)
Roll 1
0 Remain
1 Remains (Farkel)
Roll 2
0 Remain
1 Remains
2 Remain (Farkel)
Roll 1
0 Remain
1 Remains (Farkel)

Figure 4

Portions of the chart repeat themselves. Every time one starts with one die, and rolls through to the end, the flowchart looks the same (see the dashed rectangles). Further, if we could look at the case were one starts with 6 dice, this same structure would be repeated many times, as would the structure associated with rolling two dice. In fact, every time one has k dice left to roll, it looks the same as any other time one has k dice to roll. Also, these structures are always independent of what comes before. Using these facts, we can use simpler events to build up to the case where one starts with 6 dice.

Let us give names and symbols to some pieces of information. Let X denote the points earned, and let $\mu_i = E(X \mid i \text{ dice rolled})$. Then we eventually want μ_6. We will also be interested in how often one does not farkel. We shall call this $\pi_i = P(\text{not farkelling} \mid i \text{ dice rolled})$. A related statistic is the expected value of points given that one does not farkel. This shall be denoted $\varepsilon_i = E(X \mid i \text{ diced rolled and not farkelling})$. While all of these values seem similar to the E_j^i's and the P_j^i's, they are indeed different. All three of these values consider rerolling the dice until all the dice are used, until the player farkels, or until the player stops rolling. The E_j^i's and P_j^i's consider solely one roll of the dice.

So suppose that for some number of dice, all the probabilities and expected values associated with rolling fewer dice are known. For concreteness, suppose that the μ_i's, π_i's, and ε_i's are known for all values of i less than three. We also already have all the P_3^i's and E_3^i's from before. We are now interested in finding μ_3, π_3, and ε_3.

We already have a natural partition of the event "rolling three dice" with the events "no dice remain," "one die remains," "two dice remain," and "farkel." By Theorem 2, if we can find the expected value for each of those events, we can find μ_3. Once again, for concreteness sake, let us consider the event "one die remains."

We are looking for $E(X \mid 3 \text{ dice rolled and 1 remains})$ where once again this expected value considers subsequent rolls. The probability of earning any points by rolling one die is π_1. Further, we should expect to earn $E_3^1 + \varepsilon_1$ points from this: E_3^1 points just to get in the position to have one die left, and another ε_1 points from not farkelling. Summing these expectations is justified by Theorem 3. The remaining portion of the time, one farkels. Hence by Theorem 2,

$$E(X \mid 3 \text{ rolled, 1 remain}) = \pi_1(E_3^1 + \varepsilon_1) + (1 - \pi_1) \cdot 0 = \pi_1(E_3^1 + \varepsilon_1).$$

The same reasoning holds for all the remaining events except for the case when one farkels, in which case the expected point value is 0. By an application of Theorem 2, we have that

$$\begin{aligned}\mu_3 &= \sum_{i=0}^{2} P(i \text{ remain}) E(X \mid 3 \text{ rolled}, i \text{ remain}) + P(\text{farkel}) \cdot 0 \\ &= \sum_{i=0}^{2} P_3^i \pi_i (E_3^i + \varepsilon_i)\end{aligned}$$

Using the Law of Total Probability and that

$$P(\text{not farkelling} \mid 3 \text{ dice remain}) = 0,$$

we also obtain:

$$\begin{aligned}\pi_3 &= \sum_{i=0}^{3} P(i \text{ dice remain}) P(\text{not farkelling} \mid i \text{ dice remain}) \\ &= \sum_{i=0}^{2} P(i \text{ dice remain}) P(\text{not farkelling} \mid i \text{ dice remain}) \\ &= \sum_{i=0}^{2} P_3^i \pi_i\end{aligned}$$

By Theorem 2 we also have that

$$\begin{aligned}\mu_3 &= P(\text{not farkelling}) E(X \mid \text{not farkelling}) \\ &\quad + P(\text{farkelling}) E(X \mid \text{farkelling}) \\ &= \pi_3 \varepsilon_3 + 0.\end{aligned}$$

Therefore $\varepsilon_3 = \mu_3 / \pi_3$.

Hence we were able to find μ_3, π_3, and ε_3. To generalize this result, if μ_i, π_i, and ε_i are known for all $i < j$ for some $j \in \mathbb{Z}^+$, then

$$\mu_j = \sum_{i=0}^{j-1} P_j^i \pi_i (E_j^i + \varepsilon_i) \qquad \pi_j = \sum_{i=0}^{j-1} P_j^i \pi_1 \qquad \varepsilon_j = \frac{\mu_j}{\pi_j}$$

So if we can determine the bases cases of these three values, we can find μ_6, π_6, and ε_6.

The base cases are rather easy if one recalls the assumption that one will always stop rolling if there are n or fewer dice remaining. In this case, $\mu_i = \varepsilon_i = 0$ for all $i \leq n$. In those cases the player is not rolling, and they earn no points. Further, $\pi_i = 1$ for all $i \leq n$. The player is not rolling the dice, so there is no way to farkel. With these starting values, all the μ_i's, π_i's, and ε_i's are now easily found by iterating the above equations.

We now need another set of components. The first of these is the probability of using all 6 dice, and the second is the expected value of those rolls using all 6 dice. These values will determine how many points, and how often points are earned in the chain shown in Figure 2. Let

$$\phi_i = P(\text{using all dice} \mid \text{starting with } i \text{ dice}),$$

and let

$$F_i = E(X \mid \text{all dice used starting with } i \text{ dice}).$$

We want ϕ_6 and F_6.

Once again, assume that ϕ_i and F_i are known for all values of i less than some value j. We shall once again use $j = 3$ as a concrete example. ϕ_3 can be determined immediately by Theorem 1:

$$\begin{aligned}\phi_3 &= \sum_{i=1}^{3} P(i \text{ remain}) P(\text{use all dice} \mid i \text{ remain}) \\ &= \sum_{i=0}^{2} P_3^i \phi_1 + P_3^3 0 = \sum_{i=0}^{2} P_3^i \phi_i\end{aligned}$$

To find F_3, we once again consider the division induced by how many dice remain, and we shall again use the case where 1 die remains as an example. Once 1 die remains, the player expects to earn $E_3^1 + F_1$ points by Theorem 3: E_3^1 points for having 1 die remaining, and F_1 for using all the remaining dice. Further, the probability of one die remaining and using that one die, given that all the dice are being used, is $(P_3^1\phi_1)/\phi_3$. Hence, generalizing this and using Theorem 2, we have

$$F_3 = \sum_{i=0}^{2} \frac{P_3^i \phi_i}{\phi_3} (E_3^i + F_i)$$

Note that dividing by ϕ_3 is not an issue. It is always possible to use all the dice given that one rolls three dice.

Therefore for any positive integer j, if ϕ_i and F_i are known for all $i < j$, then

$$\phi_j = \sum_{i=0}^{j-1} P_j^i \phi_i \qquad F_j = \sum_{i=0}^{j-1} \frac{P_j^i \phi_i}{\phi_j} (E_j^i + F_i)$$

Once again, the base cases are easily found. With n as before, $\phi_i = 0$ for all $1 \le i \le n$ since in those cases, the player is not rolling and cannot possibly use all the dice. In the case of ϕ_0, however, all the dice have already been used, so $\phi_0 = 1$. Additionally, $F_i = 0$ for all $i \le n$, since the player is not earning any additional points.

We now only need two more items before we have our total expected value. We need the expected value of those rolls which terminate, and the probability of rolling such a hand. We shall call the former

$$G = E(X \mid \text{player stops rolling})$$

and the latter

$$\gamma = P(\text{player stops rolling}).$$

These cases occur when the player has n or fewer dice left, but not 0 dice left. Note that we can find these directly for the case when we roll 6 dice without resorting to summations and subscripts as we have done earlier.

Finding γ is particularly easy. Note that after rolling 6 dice, we know the probability of farkelling is $1 - \pi_6$, and the probability of using all the dice is ϕ_6. Since the events of farkelling, using all dice, and stopping with dice left are mutually exclusive and exhaust all possibilities, $1 = (1 - \pi_6) + \phi_6 + \gamma$. Hence $\gamma = \pi_6 - \phi_6$.

Similarly, we know that $\mu_6 = 0(1-\pi_6) + G \cdot \gamma + F_6 \cdot \phi_6$ by Theorem 2. Solving for G, we have that

$$G = \frac{\mu_6 - F_6\phi_6}{\gamma}$$

It is worth mentioning that $\gamma = 0$ iff $\pi_6 = \phi_6$. Hence if $\gamma = 0$, then the only options are farkelling and using all the dice. Such a situation never occurs in any practical application, since this would mean that one would always continue rolling until one farkels.

Now that we have all the constituent pieces, we can build our total expected value. The probability of rolling all the dice k times, followed by a farkel is $(\phi_6)^{k-1}(1 - \pi_6)$. Similarly, the probability of rolling all the dice k times and then stopping is $(\phi_6)^{k-1}\gamma$. For the first situation, we have earned 0 points. For the second situation we expect to earn $F_6(k-1) + G$ points. Hence, by Theorem 3, our total expected value is

$$\begin{aligned} E(X) &= \sum_{x=1}^{\infty}(\phi_6)^{x-1}(1-\pi_6)0 + \sum_{x=1}^{\infty}(\phi_6)^{x-1}\gamma[F_6(x-1)+G] \\ &= \sum_{x=1}^{\infty}(\phi_6)^{x-1}\gamma[F_6(x-1)+G] \end{aligned}$$

Therefore, we only need to evaluate the series

$$\sum_{x=1}^{\infty}(\phi_6)^{x-1}\gamma[F_6(x-1)+G],$$

which is easily done by programs such as *Maple* or *Mathematica*. A quick application of the ratio test shows that our series does converge.

Here is the entire algorithm in summary:

1. Compute all the P_j^i's and E_j^i's.
2. Set $\pi_i = 1$ and $\varepsilon_i = 0$ for $i \le n$.
3. Compute μ_j, π_j, and ε_j, in that order, using the equations
$$\mu_j = \sum_{i=0}^{j-1} P_j^i \pi_i (E_j^i + \varepsilon_i) \qquad \pi_j = \sum_{i=0}^{j-1} P_j^i \pi_i \qquad \varepsilon_j = \frac{\mu_j}{\pi_j}$$
for j from $n+1$ to 6.
4. Set $\phi_0 = 1$, and set $\phi_i = 0$ for $1 \le i \le n$. Also set $F_i = 0$ for $i \le n$.
5. Compute ϕ_j and F_j, in that order, for j from $n+1$ to 6 using the equations
$$\phi_j = \sum_{i=0}^{j-1} P_j^i \phi_i \qquad F_j = \sum_{i=0}^{j-1} \frac{P_j^i \phi_i}{\phi_j}(E_j^i + F_i)$$
6. Compute $\gamma = \pi_6 - \phi_6$.
7. Compute
$$G = \frac{\mu_6 - F_6 \phi_6}{\gamma}$$
8. Compute the final desired value:
$$\sum_{x=1}^{\infty} (\phi_6)^{x-1} \gamma [F_6(x-1) + G]$$
This is the total expected value of rolling 6 dice, with rerolling, and stopping with n or fewer dice.

6. Implementing the Algorithm

Despite its apparent complexity, the algorithm is actually quite easy to use in practice. For any practical use, a computer algebra system must be utilized. After the original data and the algorithm are plugged into a program such as *Maple*, the results follow almost instantly. Further, once the original data (the E_j^i's and P_j^i's) is known, modifying the data for a particular strategy takes a few minutes. Since the only thing that changes with different strategies is the E_j^i's, P_j^i's, and n, all that is required is to change a few lines in the *Maple* code.

7. Remaining Thoughts

An obvious question is how well does this algorithm work? In order to see if the results produced agreed with reality, I performed the following experiment. Using the strategy of "no strategy" (that is, any points that I rolled were kept), and stopping if I had 3 or fewer dice remaining, the algorithm calculated an expected value of 503.8 points. I then simulated 600 farkel hands. The final average score was 507.8 points. The results would seem to imply that the algorithm is accurate.

As an example of the applicability of the algorithm, consider the strategy of not keeping single 5's. Alone, a 5 is only worth 50 points. Hence, by rerolling that die, one sacrifices relatively few points in order to have a lessor probability of farkelling. Using the algorithm, it is predicted that not keeping single 5's and stopping with 3 or fewer dice remaining produces an average score of 545 points. Therefore, not keeping single 5's is advantageous.

With some basic probability theory and a little ingenuity, we have an accurate, albeit complex looking, algorithm for evaluating simple strategies for farkel. However, there are many things left to investigate. For example, how spread out is the point distributions? How many rolls does it take for the expected value to be important? Applying this algorithm to actual game play has shown that this is an important component in evaluating a strategy. The games are short enough that having an optimal long term strategy does not always overpower the "luck" of other players in the short run. Hence knowing the variance of the point distribution would be ideal.

Furthermore, more complicated strategies need to be evaluated. For example, if one made their strategic decisions based on how many points that they have already scored in the game, and how many points that have already scored in that hand, then this particular algorithm is no longer appropriate. Hence, we have only begun to discover all the properties of this seemingly innocent dice game. Despite these detractions, this algorithm is an accurate system for comparing simple strategies that is easy to use, and produces results quickly.

Acknowledgements: I would like to thank the faculty of Emporia State University for their comments, suggestions, and support in producing this paper. In particular, I want to thank Dr. Brian Hollenbeck for inspiring this project and serving as my advisor throughout its duration.

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[3] B. Wilson, *Rules of Farkle*, http://www.agileprogrammer.com/dotnetguy, July 2005.

Announcement Concerning Honor Cords

Kappa Mu Epsilon does not provide honor cords. However, a company in New York, Schoen Trimming and Cord Company, has agreed to provide honor cords in the KME colors to chapters who would like to provide cords for their graduates. The company address is Schoen Trimming and Cord Co., Inc., 151 West 25th Street, New York, NY 10001. The toll free phone number is 1-877-827-7357 and the Fax is 1-212-924-4945. The email address is Schoentrims@aol.com. Ask for #123 double honor cords and mention Kappa Mu Epsilon. The colors are rose pink and silver. The pricing is $39/dozen with $9 for shipping and handling. The minimum order is one dozen. Payment can be made by check or credit card.

The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before January 1, 2008. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Spring, 2008 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859)-622-3051)

CORRECTED AND CONTINUING PROBLEMS 600, 602
(solutions due October 1, 2007)

No or few solutions other than those of the proposers have been submitted for the following problems, so we extend the deadline for submission of a solution.

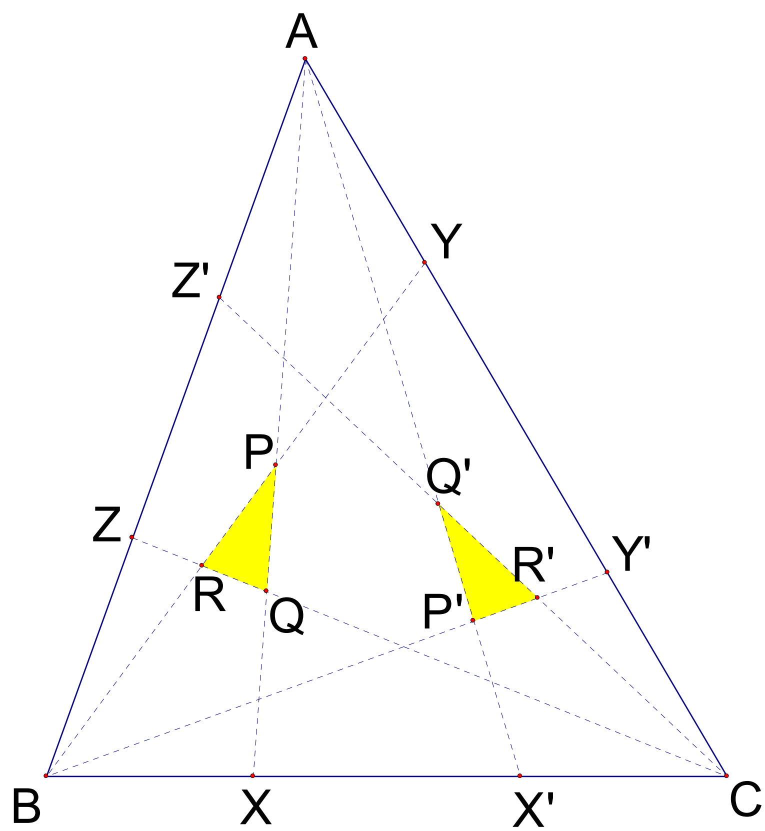
Problem 600. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA.*

In ΔABC, let X, Y, and Z be points on sides BC, CA, and AB, respectively. Let

$$x = \frac{BX}{XC},\ y = \frac{CY}{YA},\ \text{and } z = \frac{AZ}{ZB}.$$

The lines AX, BY, CZ bound a central triangle PQR. Let X', Y', and Z' be the reflections of X, Y, and Z, respectively, about the midpoints of the sides of the triangle upon which they reside. These give rise to a central triangle $P'Q'R'$. Prove that the area of ΔPQR is equal to the area of $\Delta P'Q'R'$ if and only if either

$$x = y \text{ or } y = z \text{ or } z = x.$$



Problem 602. *(Corrected) Proposed by the editor.*

Consider the sequence of polynomials recursively defined by

$$\begin{aligned} p_1(x) &= (x-2)^2 \\ p_2(x) &= [p_1(x)-2]^2 \\ &\vdots \\ p_n(x) &= [p_{n-1}(x)-2]^2 \\ &= x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_2x^2 + a_1x + 4, \end{aligned}$$

where $m = 2^n$. Find closed formulas for the coefficients a_{m-1}, a_{m-2}, a_2, a_1.

NEW PROBLEMS

Problem 611. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Find all triplets (x, y, z) of positive numbers that satisfy the system of equations:

$$\begin{cases} x^3 - 3x + \ln\left(x^2 - x - 1\right) = y \\ y^3 - 3y + \ln\left(y^2 - y - 1\right) = z \\ z^3 - 3z + \ln\left(z^2 - z - 1\right) = x \end{cases}.$$

Problem 612. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let n be a nonnegative integer. Prove that

$$\sqrt{\frac{F_n}{F_n + 2F_{n+1}}} + \sqrt{\frac{F_{n+1}}{F_{n+1} + 2F_n}} \geq 1,$$

where F_n represents the n^{th} Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.

Problem 613. *Proposed by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.*

A point P is moving on a quarter circle of center O which is bounded by two points A and B. Let PQ be the perpendicular from P to the radius OA. The point M is chosen on the ray OP such that the length of $OM =$ length of OQ + length of QP. The N be a point on the radius OP such that $ON = OQ$. Show that the center of the locus of points M as P moves along the quarter circle is located on the locus of the points N.

Problem 614 *Proposed by the editor.*

Let $\tau(n)$ represent the number of divisors of n. For example $\tau(10) = 4$ because $1, 2, 5, 10$ are the divisors of 10. Let $\sigma(n)$ represent the sum of the divisors of n. For example, $\sigma(10) = 1 + 2 + 5 + 10 = 18$. Prove that the infinite sum $\sum_{n=1}^{\infty} \frac{4^{\tau(n)}}{5^{\sigma(n)}}$ is bounded above by the fraction $\frac{364}{375}$.

Problem 615. *Proposed by the editor.*

The sequence $a_1, a_2, a_3, \ldots$ is a monotone increasing sequence of natural numbers. It is known for any k that $a_{a_k} = 3k$. Find a formula for a_k and find the particular value a_{2007}.

SOLUTIONS 585, 589, 597-599, 601, 603

Problem 585. *(Corrected) Proposed by José Luis Diaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Solution *by the proposer was published in the Fall 2006 issue. Before the Fall 2006 issue was printed, this problem was also solved by Harrison Potter, (student), Marietta College, Marietta, OH and the Missouri State Problem Solving Group.*

Problem 589. *Proposed by Ken Wilke.*

Solution *by the proposer was published in the Fall 2006 issue. There are four solutions. They are 187248723, 387268723, 687298723, 987228713. Before the Fall 2006 issue was printed, this problem was also solved by Emily Elder (student), Slippery Rock University, Slippery Rock, PA and the Missouri State Problem Solving Group. Three solutions were found by Harrison Potter, (student), Marietta College, Marietta, OH. One solution was found by Matthew Dawson (student), Union University, Jackson, TN.*

Problem 597. *Proposed by Bangteng Xu, Eastern Kentucky University, Richmond, KY.*

Determine the following limit.

$$\lim_{n\to\infty}\frac{1n+3\,(n-1)+5\,(n-2)+\cdots+(2n-3)\cdot 2+(2n-1)\cdot 1}{n^3}$$

Solution *by Matthew Dawson (student), Union University, Jackson, TN.*

$$\begin{aligned}
& \lim_{n\to\infty}\frac{1n+3\,(n-1)+5\,(n-2)+\cdots+(2n-3)\,2+(2n-1)\,1}{n^3} \\
= & \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}(2k+1)\,(n-k)}{n^3} = \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}\left(2kn+n-2k^2-k\right)}{n^3} \\
= & \lim_{n\to\infty}\frac{n\sum_{k=0}^{n-1}1+(2n-1)\sum_{k=0}^{n-1}k-2\sum_{k=0}^{n-1}k^2}{n^3} \\
= & \lim_{n\to\infty}\frac{n^2+(2n-1)\,(n-1)\,n/2-2\,(n-1)\,n\,[2\,(n-1)+1]\,/6}{n^3}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{n\left[6n+3\left(2n-1\right)\left(n-1\right)-2\left(n-1\right)\left(2n-1\right)\right]}{6n^3} \\
&= \lim_{n\to\infty} \frac{n\left[6n+\left(2n-1\right)\left(n-1\right)\left(3-2\right)\right]}{6n^3} \\
&= \lim_{n\to\infty} \frac{n\left(6n+2n^2-3n+1\right)}{6n^3} \\
&= \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} = \frac{2}{6} = \frac{1}{3}.
\end{aligned}$$

Also solved by Harrison Potter, (student), Marietta College, Marietta, OH; the Missouri State Problem Solving Group; and the proposer.

Problem 598. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA.*

Let C be the unit circle centered at the point $(3, 4)$. Let $O = (0, 0)$ and let $A = (1, 0)$. Let P be a variable point on C, and let $PA = a$ and $PO = b$. Find a non-constant polynomial $f(x, y)$ such that $f(a, b) = 0$ for all points P on C.

Solution *by the proposer.*

If P has coordinates (x, y), we have the three equations

$$\begin{cases} (x-3)^2 + (y-4)^2 = 1 \\ x^2 + y^2 = b^2 \\ (x-1)^2 + y^2 = a^2. \end{cases}$$

Now we eliminate the variables x and y from these 3 equations. *Mathematica* will do this and give $25a^4 - 44a^2b^2 + 20b^4 + 94a^2 - 116b^2 + 457 = 0$. So the desired polynomial is $f(x, y) = 25x^4 - 44x^2y^2 + 20y^4 + 94x^2 - 116y^2 + 457$.

Also solved by Matthew Dawson (student), Union University, Jackson, TN and Harrison Potter (student), Marietta College, Marietta, OH.

Problem 599. *Proposed by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.*

Primes of the form $3n^2 + 3n + 1$ are called *Cuban primes*. Find necessary and sufficient conditions for $3n^2 + 3n + 1$ to be divisible by 7.

Solution *by Emily Elder (student), Slippery Rock University, Slippery*

Rock, PA.

To begin, we can rewrite $3n^2+3n+1$ as $3n^2+3n-6+7$. [The notation $a \mid b$ means "a divides b."] Since $7 \mid 7$, then $7 \mid 3n^2+3n+1$ if and only if $7 \mid 3n^2+3n-6$, that is, if and only if $7 \mid 3(n+2)(n-1)$. Since 7 does not divide 3, $7 \mid n+2$ or $7 \mid n-1$. Suppose $7 \mid n+2$. Then there exists an integer k such that $7k = n+2$, so that $n = 7k-2$. Similarly, if $7 \mid n-1$, then there exists an integer k such that $7k = n-1$, so that $n = 7k+1$. Thus, the necessary and sufficient conditions for $3n^2+3n+1$ to be divisible by 7 are for n to be of the form $7k-2$ or $7k+1$ for some integer k.

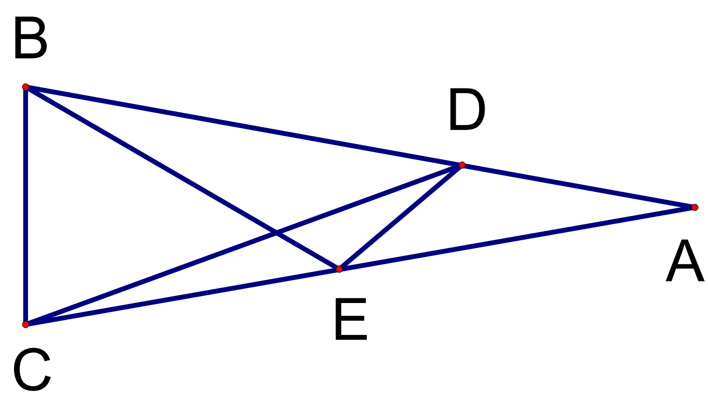
Also solved by Matthew Dawson (student), Union University, Jackson, TN; Harrison Potter (student), Marietta College, Marietta, OH; the Missouri State Problem Solving Group; and the proposers.

Problem 601. *Proposed by Johannas Winterink.*

You are given the following information about the drawn triangle:

- Point A, D, and B are collinear;
- Points A, E, and C are collinear;
- $\angle DAE = 20°$, $\angle ADE = 130°$, $\angle AEB = 140°$, $\angle ADC = 150°$.

Prove that $AB = AC$.



Solution *(jointly) by Kayleigh Bush (student), Peters Township HIgh School, and Ruowang Li (student), Waynesburng College, Waynesburg, PA.*

It is easy to show that

$$\begin{aligned}\angle AED &= 30^\circ, \angle DEG = 110^\circ, \angle DGE = 50^\circ, \angle EDG = 20^\circ,\\ \angle DGB &= 130^\circ, \angle DBG = 20^\circ, \angle EGC = 130^\circ, \text{ and } \angle ECG = 10^\circ.\end{aligned}$$

Apply the Law of Sines to the following triangles.

ΔDAE	$\dfrac{DE}{\sin 20^\circ} = \dfrac{AE}{\sin 130^\circ}$	ΔDAE	$\dfrac{AD}{\sin 30^\circ} = \dfrac{AE}{\sin 130^\circ}$
ΔDEG	$\dfrac{DG}{\sin 110^\circ} = \dfrac{DE}{\sin 50^\circ}$	ΔDEG	$\dfrac{EG}{\sin 20^\circ} = \dfrac{DE}{\sin 50^\circ}$
ΔBDG	$\dfrac{DB}{\sin 130^\circ} = \dfrac{DG}{\sin 20^\circ}$	ΔEGC	$\dfrac{EC}{\sin 130^\circ} = \dfrac{EG}{\sin 10^\circ}$

Thus,

$$\begin{aligned}
& DB + AD - AE - EC \\
= & \left(\frac{\sin 130^\circ}{\sin 20^\circ}\right) DG + \left(\frac{\sin 30^\circ}{\sin 130^\circ}\right) AE - AE - \left(\frac{\sin 130^\circ}{\sin 10^\circ}\right) EG \\
= & \left(\frac{\sin 130^\circ}{\sin 20^\circ}\right)\left(\frac{\sin 110^\circ}{\sin 50^\circ}\right) DE + \left(\frac{\sin 30^\circ}{\sin 130^\circ}\right) AE \\
& -AE - \left(\frac{\sin 130^\circ}{\sin 10^\circ}\right)\left(\frac{\sin 20^\circ}{\sin 50^\circ}\right) DE \\
= & \left(\frac{\sin 130^\circ}{\sin 20^\circ}\right)\left(\frac{\sin 110^\circ}{\sin 50^\circ}\right)\left(\frac{\sin 20^\circ}{\sin 130^\circ}\right) AE + \left(\frac{\sin 30^\circ}{\sin 130^\circ}\right) AE \\
& -AE - \left(\frac{\sin 130^\circ}{\sin 10^\circ}\right)\left(\frac{\sin 20^\circ}{\sin 50^\circ}\right)\left(\frac{\sin 20^\circ}{\sin 130^\circ}\right) AE \\
= & AE\left(\frac{\sin 110^\circ}{\sin 50^\circ} + \frac{\sin 30^\circ}{\sin 130^\circ} - 1 - \frac{\sin^2 20^\circ}{\sin 10^\circ \sin 50^\circ}\right) \\
= & AE\left(\frac{\sin 70^\circ}{\sin 50^\circ} + \frac{\sin 30^\circ}{\sin 50^\circ} - 1 - \frac{\sin^2 20^\circ}{\sin 10^\circ \sin 50^\circ}\right) \\
= & AE\left(\frac{\sin 70^\circ}{\sin 50^\circ} + \frac{\sin 30^\circ}{\sin 50^\circ} - 1 - \frac{4\sin^2 10^\circ \cos^2 10^\circ}{\sin 10^\circ \sin 50^\circ}\right) \\
= & AE\left(\frac{\sin 70^\circ}{\sin 50^\circ} + \frac{\sin 30^\circ}{\sin 50^\circ} - 1 - \frac{4\sin 10^\circ \cos^2 10^\circ}{\sin 50^\circ}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{AE}{\sin 50^\circ}\right)\left[\sin 70^\circ + \sin 30^\circ - \sin 50^\circ - 4\sin 10^\circ \left(1 - \sin^2 10^\circ\right)\right] \\
&= \left(\frac{AE}{\sin 50^\circ}\right)\left(\sin 70^\circ + \sin 30^\circ - \sin 50^\circ - 4\sin 10^\circ + 4\sin^3 10^\circ\right) \\
&= \left(\frac{AE}{\sin 50^\circ}\right)(\sin 70^\circ + \sin 30^\circ - \sin 50^\circ - 4\sin 10^\circ \\
&\quad +3\sin 10^\circ - \sin 30^\circ) \\
&\quad \text{from } 4\sin^3\theta = 3\sin\theta - \sin(3\theta) \\
&= \left(\frac{AE}{\sin 50^\circ}\right)(\sin 70^\circ - \sin 50^\circ - \sin 10^\circ) \\
&= \left(\frac{AE}{\sin 50^\circ}\right)[\sin(60^\circ + 10^\circ) - \sin(60^\circ - 10^\circ) - \sin 10^\circ] \\
&= \left(\frac{AE}{\sin 50^\circ}\right)(2\cos 60^\circ \sin 10^\circ - \sin 10^\circ) \\
&= \left(\frac{AE}{\sin 50^\circ}\right)(\sin 10^\circ - \sin 10^\circ) \\
&= 0.
\end{aligned}
$$

So $DB + AD = AE + EC$ and $AB = AC$.

Also solved by the proposer.

Problem 603. *Proposed by the editor.*

Consider the following variant on Pascal's triangle. Start with the top two rows the same as in Pascal's triangle. For the remaining rows, put 1 at each end. For each interior entry, add the two diagonal values above the position plus the value in the row above which is between the two summed values. This means an interior entry is a sum of values in the equilateral triangle above the position. Rows 0, 1, 2, 3, and 4 of the triangle are:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 3 & & 1 & & \\
 & 1 & & 5 & & 5 & & 1 & \\
1 & & 7 & & 13 & & 7 & & 1
\end{array}.
$$

One notable fact about this triangle is that all entries in the triangle are odd. Another fact is that the second diagonal is the set of odd numbers. Find a closed formula for the sum of the entries across the m^{th} row.

Solution *by Harrison Potter (student), Marietta College, Marietta, OH.*

Let r_n be the sum of the entries in the n^{th} row. Thus, $r_0 = 1$, $r_1 = 2$, $r_2 = 5, \ldots.$ From the entries directly above the numbers in the n^{th} row comes a contribution to r_n of r_{n-2}. From the entries diagonally above the numbers in the n^{th} row comes a contribution of 2 times interior values and the ones at each end. Doubling the ones at each end will give the new ones on the end of the n^{th} row. So $r_n = 2r_{n-1} + r_{n-2}$. Using $r_n = r_n$, we arrive at a matrix equation

$$\left[\begin{array}{c} r_{n+1} \\ r_n \end{array}\right] = \left[\begin{array}{cc} 2 & 1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{c} r_n \\ r_{n-1} \end{array}\right].$$

Let $A = \left[\begin{array}{cc} 2 & 1 \\ 1 & 0 \end{array}\right]$. Then applying this equation to itself repeatedly until the column vector on the right is smallest, we get

$$\left[\begin{array}{c} r_{n+1} \\ r_n \end{array}\right] = A^n \left[\begin{array}{c} r_1 \\ r_0 \end{array}\right] = A^n \left[\begin{array}{c} 2 \\ 1 \end{array}\right].$$

We can find a formula using the eigenvalues of the matrix A. The characteristic polynomial of A is $C_A(x) = x^2 - 2x - 1$. Thus A has eigenvalues $\lambda_1 = 1 + \sqrt{2}$ and $\lambda_2 = 1 - \sqrt{2}$. Then $r_n = 2a_1 + a_0$, where

$$a_1 = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \text{ and } a_0 = \frac{\lambda_1 \lambda_2^n - \lambda_1^n \lambda_2}{\lambda_1 - \lambda_2}.$$

So

$$\begin{aligned} r_n &= \frac{1}{\lambda_1 - \lambda_2}\left(2\lambda_1^n - 2\lambda_2^n + \lambda_1\lambda_2^n - \lambda_1^n\lambda_2\right) \\ &= \frac{1}{2\sqrt{2}}\left[\lambda_1^n\left(2 - \lambda_2\right) + \lambda_2^n\left(\lambda_1 - 2\right)\right] \\ &= \frac{1}{2\sqrt{2}}\left[\lambda_1^n\left(\lambda_1\right) + \lambda_2^n\left(-\lambda_2\right)\right] \\ &= \frac{1}{2\sqrt{2}}\left(\lambda_1^{n+1} - \lambda_2^{n+1}\right) \\ &= \frac{1}{2\sqrt{2}}\left[\left(1 + \sqrt{2}\right)^{n+1} - \left(1 - \sqrt{2}\right)^{n+1}\right]. \end{aligned}$$

Also solved by the proposer.

Kappa Mu Epsilon News

Edited by Connie Schrock, Historian

Updated information as of January 2007

Send news of chapter activities and other noteworthy KME events to

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Chapter News

AL Alpha – Athens State University

Chapter President– Mariel Gray, 20 Current Members, 15 New Members
Other fall 2006 officers: Allison Stanford, Vice–President; Nick Retherford, Secretary; Meaghan Mitchell, Treasurer; Dottie Gasbarro, Corresponding Secretary.

During the fall 2006 semester at Athens State University, Alabama Alpha chapter participated in two service projects and held one meeting. Professor Beth Allen was the guest speaker at the September meeting and spoke about Geo Caching. She had several activities that attendees participated in and several students "found" the hidden treasure!

Members worked in the KME and MACS club (Math And Computer Science club) food booth at the Old Time Fiddler's Convention held annually on the first weekend in October on the historic Athens State University campus in Athens, AL. KME and MACS members, alumni, and faculty cooked and sold hamburgers, hotdogs and all the Southern fixin's raising over $1000 for travel and/or conference scholarships for Math and Computer Science students. KME members also collected toys, clothes, and necessities for Operation Santa Claus during November and December, providing Christmas items for needy families in our community.

Initiation of New Members will be held in April.

AL Gamma – Montevallo University

Don Alexander, Corresponding Secretary.

New Initiates – Mary Margaret Clapp, Amber Wright, Jessica Tischler, Dennis Hall II, Sabrina Mims, Jessica Langevin, John Herron, Krystle Ames, Aleah Gothard, April Huggins, Sarah Robinson, MeCherri Traver, Lauren M. Weil.

AL Zeta – Birmingham Southern College

Chapter President– Gardner Moseley, 11 Current Members, 5 New Members

Other fall 2006 officers: Kelly Bragan, Vice–President; David Ray, Secretary; Jill Stupiansky, Treasurer; Mary Jane Turner, Corresponding Secretary.

New Initiates – Jack Guy DaSilva, Brittany Diane Green, Jason Michael Gruber, John Robert Monk, John William Padley II.

CA Epsilon – California Baptist University

Jim Buchholz, Corresponding Secretary.

New Initiates – Matthew Brown, Jamie Griffitts, Sarah Gwilt, Urs Gunthor, Jeff Heinz, Jonathan Hines, Brett Sanchez, Armando Serrano.

CO Delta – Mesa State College

Erik Packard, Corresponding Secretary.

New Initiates – Michael D. Brooks, Eric W. Miles, Desarae L. Moots, Kyle W. Rozean, Austin H. Schneider, Matthew J. Seymour.

CO Gamma – Fort Lewis College

Deborah Berrier, Corresponding Secretary.

New Initiates – Natalie Eich, Chiharu Fujii, Heidi Hendricks, James Jones, Jamie George, Jeff Gjere, Shaemus Gleason, Joanna Gordon, Dan Graybill, Christopher Morris, Kristoffer Persson, Don Sohis, Alisha Gwen Swanson, Peggy Vorald.

CT Beta – Eastern Connecticut State University

Fall 2006 officers: Mizan R. Khan, Treasurer; Christian L. Yankov, Corresponding Secretary.

FL Beta – Florida Southern College

Allen Wuertz, Corresponding Secretary.

New Initiates – Samantha Joan Bethel, Ian Matthew Johnson, Allison B. Mitchell, Gwendolyn H. Walton.

IA Alpha – University of Northern Iowa

Chapter President– Paul Grammens, 38 members, 4 New Members.

Other fall 2006 officers: Jake Ferguson, Vice–President; Erin Conrad, Secretary; Brenda Funke, Treasurer; Mark D. Ecker, Corresponding Secretary.

Our first Fall KME meeting was held on September 19, 2006 at Professor Mark Ecker's residence and the University of Northern Iowa Homecoming Coffee was held at Professor Suzanne Riehl's residence on

October 7, 2006. Our second meeting was held on October 18, 2006 at Professor Russ Campbell's residence where student member Colby Goetsch talked about his work estimating medical cost trends at Aetna the previous summer. Our third meeting was held on November 14, 2006 at Professor Jerry Ridenhour's residence where student member Bill Freese presented his paper on "Measurement of the Earth in Ancient Times". Student member Brenda Funke addressed the fall initiation banquet with "The Murphy's Law Phenomenon". Our Fall banquet was held at Godfather's restaurant in Cedar Falls on December 5, 2006 where four new members were initiated.

New Initiates – Emily Blad, Joe Decker, Andy Quint, Adam Schneberger.

IA Delta – Wartburg College

Chapter President– Justin Peters. 24 Current Members, 0 New Members

Other fall 2006 officers: Joee Williams, Vice–President; Jill Seeba, Secretary; Tim Schwickerath, Treasurer; Dr. Brian Birgen, Corresponding Secretary.

At the Wartburg Homecoming Renaissance Fair, our club successfully ran our traditional annual fundraiser by selling egg-cheeses. We sponsored a field trip down to Kansas City to see an original copy of Isaac Newton's Principia.

New Initiates – Sagar Khushalani, David Kordahl, David Neil, Kevin Schreader, Timothy Schwickerath, Prateek Shrestha, Tyler Vachta, Jeffrey Zittergruen.

IL Beta – Eastern Illinois University

Andrew Mertz, Corresponding Secretary.

New Initiates – Rick Anderson, Holly Bertram, Jennifer Muser, David Cesar, Matthew Niemerg, Doug Cichon, Stephen Puricelli, Kari Sue Donoho, Carol Ann Reuscher, Adam "Josh" Due, Amber Schmidt, Jonathan Hood, Vincent Shamhart.

IL Eta – Western Illinois University

Boris Petracovici, Corresponding Secretary.

New Initiates – Christopher Barenz, Sarah Cane, Tara DeMay, Sarah Hays, Stephanie Heaton, Breanne Hoffman, Jennifer Newberg, Dennis Norton.

IL Theta – Benedictine University

Chapter President – Jennifer Muskovin, 15 Current Members, 0 New Members

Other fall 2006 officers: Brad Callard, Vice-President; Debra Witczak, Secretary; Lisa Townsley, Corresponding Secretary.

During the fall, the students organized: a calculus competition and a chess competition. They volunteered to assist the student government at a poker night. They rallied other students to attend our guest speaker in mathematical biology—over 150 students were present.

IL Zeta – Dominican University

Marion Weedermann, Corresponding Secretary.

New Initiates – Yoana Azmanova, Catherine Calixto, Teresamarie Cervone, Christopher Gallicchio, Stephanie Majkowicz, Kristen McNamara, Stephanie Orchard, Ryan Riske, Isaac Shamoon, Malissa Wegener

IN Alpha – Manchester College

Stanley Beery, Corresponding Secretary.

New Initiate – Georgi Chkunev.

IN Beta – Butler University

Chapter President– Laura Laycok, 22 Current Members, 7 New Members
Other fall 2006 officers: Taryn Schmidt, Vice–President; James Schuster, Secretary; Keenan Hecht, Treasurer; Amos Carpenter, Corresponding Secretary.

In addition to our monthly meetings we brought two invited speakers to campus. Dr. Rich Stankewitz, Graduate Program Director at Ball State University, Muncie, Indiana, presented Chaos Theory – Real and Complex Dynamics. Dr. David Groggel, Associate Professor of Statistics at Miami University, Oxford, Ohio, presented Streaks in Sports.

New Initiates – Daisy A. Chew, Weston K. Edens, Brent R. Freed, Whitney K. Lucas, Lindsey H. Pattern, Cora A Pauli, Matthew J Schonauer.

KS Alpha – Pittsburg State University

Chapter President – Erin Wells, 34 Current Members, 7 New Members
Other fall 2006 officers: Dusty Peterson, Vice-President; Casey Kuhn, Secretary; John Cauthon, Treasurer; Dr. Tim Flood, Corresponding Secretary.

Casey Kuhn, mathematics education major, spoke about her experience at a summer math research “camp”. Dr. Bobby Winters presented “Redneck Mathematics”. Dr. Cynthia Woodburn presentation on Sudoku puzzles and variations of Sudoku puzzles.

New Initiates – Emily Brown, Morgan Brown, Michael Eaton, James Ira Moore, Benjamin Naumann, Jelinda Smith, Tosha Terveen.

KS Beta – Emporia State University

Chapter President– Mike Moore, 26 Current Members, 5 New Members
Other fall 2006 officers: Cori Samskey, Vice–President; Debbie Bolen, Secretary; Jarrett Leeds, Treasurer; Connie Schrock, Corresponding Secretary.

KS Beta chapter held a calculator workshop for algebra students. We also hosted a Math Jeopardy and participated in Math Day. Several presentations were held throughout the semester a few of them included “Math in the Movies” by Dr. Charlie Smith from Park University and “Sudoku” by Dr. Cynthia Woodburn from Pittsburg State University.

KS Delta –Washburn University

Chapter President– Kristin Ranum, 30 Current Members, 0 New Members
Other fall 2006 officers: Tammy Bolen, Vice–President; Fai Ng, Secretary; Fai Ng, Treasurer; Kevin Charlwood, Corresponding Secretary.

During the Fall semester, our KME chapter had three luncheon meetings with our math club, Club Mathematica. We hosted a former graduate who teaches middle school locally, and he gave a presentation on what his teaching position is like. Two of our students are preparing KME projects for presentation at the KME national meeting coming up in April 2007 in Springfield, Missouri.

KS Epsilon – Fort Hays State University

Jeffrey Sadler, Corresponding Secretary.

New Initiates – Roger Bach, Ann Brungardt, Jerome Conner, Jeremy Danler, Kyndra Dobson, Joan Dreiling, Charles Hansen, James Hauch, Kristy Koch, Jacqueline McDowell, Brandon Nimz, Aubrey Rankin, Lance M. Sharp, Todd Sherman, Lianju Wang, Matthew Wood, Nick Packauskas.

KS Gamma – Benedictine College

Chapter President – Chris G'Sell , 4 Current Members, 0 New Members
Other fall 2006 officers: Erica Goedken , Vice-President; Josie Villa, Secretary; Dr. Linda Herndom, Corresponding Secretary.

The Kansas Gamma Chapter held their traditional Christmas wassail party at an open house in the Department of Mathematics and Computer Science. Many stopped by on a cold afternoon to enjoy the wassail and other Christmas goodies.

KY Alpha – Eastern Kentucky University

Pat Costello, Corresponding Secretary.

New Initiates – Samuel M. Bailie, Brittany D. Barger, Sarah C. Elliott, Jacob A. Held, Christina L. Hidenrite, Susan K. Malkowski, Marci R. Nash, Chadwick D. Denny, Amanda M. Glover, Brittany L. Hensley, Yongbok Lee, Sarah N. Morris, Kristina L. Newman, Michael C. Osborne, Ernest L. Presher II, Stacey L. White, Ryan C. Waldroup, Lori A. Young.

KY Beta – University of the Cumberlands

Chapter President- Sarah Strunk, 30 Current Members, 0 New Members
Other fall 2006 officers: Lane Royer, Vice–President; John Steely, Secretary; Charle Delph, Treasurer; Jonathan Ramey, Corresponding Secretary.

On September 7, the Kentucky Beta chapter helped to host an ice cream party for the freshmen math and physics majors. Along with the Mathematics and Physics Club and Sigma Pi Sigma, the chapter had a chili supper on October 12. On December 7, the entire department, including the Math and Physics Club, the Kentucky Beta chapter, and Sigma Pi

Sigma had a Christmas party with 31 people in attendance.

Dr. Reid Davis, Laurie Anderson, Charle Delph, Rebecca Engle, John Steely, Erin Newell, Katie Ruf, Shelly Schnee

MD Alpha – College of Notre Dame of Maryland

Chapter President – Kim Wall, 14 Current Members, 0 New Members
Other fall 2006 officers: Neeraj Sharma, Vice-President; Nicole Kotulak, Secretary; Vera Ulanowicz, Treasurer; Dr. Margaret Sullivan, Corresponding Secretary.

In the Fall 2006 semester, the Hypatian Society in which our KME chapter is embedded offered a twice weekly tutoring opportunity for interested students. At the monthly meeting, the members engaged in origami and tangram activities. With the Chemistry Club, we co-sponsored a movie night featuring A Beautiful Mind.

New Initiates – Karolyn Ashley Burley, Jennifer Ebert, Nicole Eigenbrode, Karie Jean Harry, Emily Siberholz, Laura Turner.

MD Beta – McDaniel College

Chapter President – Alison Bradley, 11 Current Members, 19 New Members
Other fall 2006 officers: Ashley Baker, Vice-President; Alli Biggs, Secretary; Amy Watson, Treasurer; Dr. Harry Rosenzweig, Corresponding Secretary.

During this past semester, we inducted six new students and two new faculty members. At the induction ceremony, new faculty member Italo Simonelli gave a talk on Probabilistic Number Theory. Later in the semester, Kevin McIntyre of the Economics Departments gave a talk on The Mathematics Used in Economic Models.

New Initiates – Merrick L. Brown, Latisha N. Buford, Shaqnnan Jackson, David Justus, Wesley E. Mann, Lydia D. Tomajko.

MD Delta – Frostburg State University

Chapter President – Timothy Smith, 22 Current Members, 0 New Members.
Other fall 2006 officers: Kyle Conroy, Vice-President; Nicole Garber, Secretary; Bradley Yoder, Treasurer; Dr. Mark Hughes, Corresponding Secretary.

The Maryland Delta Chapter started the semester with a meeting in mid-September where we planned our participation in a “majors fair” held in the student center. The idea was to introduce new students to the various majors and student organizations present on campus and our members represented the Department of Mathematics and KME. Displays and multimedia presentations were prepared during our meeting and the fair went very nicely. During our October meeting, we viewed a video from PBS entitled “A Mathematical Mystery Tour” concerning interesting and difficult problems of modern mathematics. Dr. Mark Hughes presented a lecture during the November meeting on Johann Bernoulli's solution of the Brachistochrone Problem.

MD Epsilon – Villa Julie College

Chapter President – Richard Haney, 23 Current Members, 20 New Members

Other fall 2006 officers: Steven Mrozinski, Vice-President; Courtney Naff, Secretary; Emily Clemens, Treasurer; Dr. Christopher E. Barat, Corresponding Secretary.

On 10/14/06, at the Chapter's second annual initiation ceremony, 15 students and 5 faculty members were initiated into the Chapter. The guest speaker for the ceremony was Dr. James Lightner, faculty member emeritus at McDaniel College and a past national officer of KME. Activities planned for the spring semester include a fund-raising raffle of computer equipment and a program of speakers, including VJC alumni, to celebrate Mathematics Awareness Month.

New Initiates – Ms. Joan Beemer, Stephen Brower, Emily Clemens, Chanel Cottman, Joanna Duckworth, Mr. Robert Garbacik, Aaron Kuhn, Steven Mrozinski, Courtney Naff, Jonathon Englebrecht, Thomas Franklin, Deepti Patel, Ms. Vallory Shearer, Dr. Susan Slattery, Wesley Smith, Dr. Janet Thiel, Brittny Thompson, Matthew Tomney, Amy Walsh

MS Alpha – Mississippi University for Women

Chapter President – Johnatan Dillon, 13 Current Members, 0 New Member

Other fall 2006 officers: May Hawkins, Vice-President; David Wages, Secretary; Vasile (Johnny) Bratan, Treasurer; Dr. Shaochen Yang, Corresponding Secretary.

Two meetings were held, and at one of the meetings three shoe boxes of Christmas presents for "Operation Christmas Child".

MS Delta – William Carey College

Charlotte McShea, Corresponding Secretary.

New Initiates – Tim Brown, Malissa Flowers, Kristy Thurman, Summer Housley, Christopher Knight, Elizabeth Cook, Karen Embry, Katie Gardner, Jenny Guidroz, Daniel McShea, Jesse Colton Smith, Rachel Whitehead, Anthony Williams Jr., Michelle Buckley, Ashlee Britt, Elizabeth McShea, Lisa Smith.

MS Gamma – University of Southern Mississippi

Jose N. Contreras, Corresponding Secretary.

New Initiates – Amber Alderman, Sarah Buford, Amber Barnes, Chaz Ladner, Carol Shree Roberts, Khue D. Nguyen.

MO Alpha – Missouri State University

Chapter President– Uriah Williams, 24 Current Members, 7 New Members Other fall 2006 officers: Megan Reineke, Vice–President; Annie Johnson, Secretary; Thomas Buck, Treasurer; John Kubicek, Corresponding Secretary.

The Missouri Alpha Chapter of Kappa Mu Epsilon hosted the Fall Mathematics Department Picnic and held three monthly meetings. Two faculty members and two students made presentations at the monthly meeting. Dr. Kishor Shah spoke on "Women in Mathematics." Dr Kanghui Guo spoke on "Various Summation Methods." Megan Reineke spoke on Buffon's Needle Problem and Extensions." Benjamin Hill spoke on "Uniformly Convergent Series."

New Initiates – John J Garner, Christina Enneking, Benjamin Hill, Chris Inabnit, Kimberly Moss, Travis Singleton, Chris Trivitt.

MO Beta – Central Missouri State University

Rhonda McKee, Corresponding Secretary.

New Initiates – Sandy Davidson, Georgia Dunlap, Abby Rausch.

MO Epsilon – Central Methodist University

Linda O. Lembke, Corresponding Secretary.

New Initiates – Tonya Goosen, Erin Valentine, Ross Asbury, Jennifer Lester.

MO Eta –Truman State University

Jason Miller, Corresponding Secretary.

New Initiates – Alan C. Schrader, Tony Lam, Kensey L. Riley, Matthew J. Sealy, David M. Failing, Amanda K. Hamilton, April E. Sommer, Katie N. Evans, Adam C. Gouge, David A. Kiblinger, Aubrie J. Hackathorn, Nirjal Sapkota.

MO Gamma – William Jewell College

Chapter President– Andrew Gard, 14 Current Members, 0 New Memebers Other fall 2006 officers: Elizabeth Jones, Vice–President; Cameron Cupp, Secretary; Dr. Mayumi Sakata Derendinger, Treasurer; Dr. Mayumi Sakata Derendinger, Corresponding Secretary.

MO Iota – Missouri Southern State University

Chapter President – Ben Cartmill, 10 Current Members, 0 New Members Other fall 2006 officer: Rikki McCullough, Vice-President; David Smith, Secretary; Chip Curtis, Corresponding Secretary.

The chapter held monthly meetings, one of which included a presentation on applications of mathematics to finance by faculty member Dr. Yuanjin Liu. Chapter members cooked and served food at the concession stands for the home football games. In November, the chapter bought a Thanksgiving meal for a local family and in December outfitted a local 4th grade classroom with supplies. The chapter was awarded 2nd Place in a campus-wide gingerbread house contest.

MO MU – Harris Stowe State College

J. Behle, Corresponding Secretary.

This fall we held a mostly social meeting. We also presented a problem concerning the focal point of a parabolic mirror formed by lining the interior of an umbrella with aluminum foil. We were attempting determine where the sun would be focused by the parabolic shape and intended to measure the temperature at that point.

MO Nu – Columbia College

Chapter President – Heidi Steenblock, 15 Current Members, 0 New Members

Other fall 2006 officers: Mandy Jorgenson, Vice-President; Chris Schoonover, Treasurer; Dr. Ann Bledsoe, Corresponding Secretary.

KME members had worked on several projects during the fall semester 2006: they upgraded a wallet size tip tables and posted them on the KME bulletin board; volunteered at the Ronal McDonald House (prepared and served hot meals there).

MO Theta – Evangel University

Chapter President– Joshua Thomassen, 14 Current Members, 0 New Members.

Other fall 2006 officers: Lurena Erickson, Vice–President; Don Tosh, Corresponding Secretary.

Meetings were held monthly. The president, Liz Hereth, graduated early and did not return for the fall semester. So the vice-president, Josh Thomassen, became president and Lurena Erickson was elected as the new vice president. The final meeting was an ice cream social held at Dr. Tosh's house.

NE Beta – University of Nebraska at Kearney

Current President – Adam Haussler, 16 Current Members, 3 New Members

Other fall 2006 officer: John Auwerda, Vice-President; Abby Om, Secretary; Adam Sevenkar, Treasurer; Dr. Katherine Kime, Corresponding Secretary.

Graduating KME members Michael Bachman and Carrie Divis were honored by the College of Natural and Social Sciences prior to Fall Commencement. Michael has taken a position as a financial officer at Farm Credit Services of America in Grand Island, Nebraska and Carrie will be teaching in the Omaha/Lincoln area. In September, KME had a table at Mardi Gras. A new t-shirt was developed, with special effort and attention by Neil Hammond, former president who will be graduating in Spring 2007 after his student teaching.

New Initiates – David W. Aufrecht, Amber Nabity, Sasha Anderson.

NE Delta – Nebraska Wesleyan University

Chapter President– Marcus Hatfield, 13 Current Members, 0 New Members

Other fall 2006 officers: Zach Brightweiser, Vice President; Kyle Nelson, Secretary; Melissa Erdmann, Corresponding Secretary.

NE Gamma – Chadron State College

Dr. Robert Stack, Corresponding Secretary.

New Initiates – Shari Miller, John Strand, Pamela Anderson, Tyler Bartlett, Loni Hughes, Joe McLain, Leslie Mueller.

NJ Gamma – Monmouth University

Chapter President – Krystle Hinds, 20 Current Members, 9 New Members

Other fall 2006 officers: Debra Cagliostro, Vice-President; Meghan Moratelli, Secretary; Jill Banholzer, Treasurer; Jennifer Sloan, Historian; Jennifer Kroh and Leslie Cordasco, Student Liasons; Judy Toubin, Corresponding Secretary.

On Oct. 13, 2006, we held our 2nd annual volleyball game between faculty and students. The KME officers held monthly meetings and co-sponsored a colloquium held on November 8. The colloquium was directed towards undergraduates interested in math. A statement for the Math Department newsletter was submitted.

NY Iota – Wagner College

Dr. Zohreh Shahvar, Corresponding Secretary.

New Initiates – Christine Wendt, Richard A. Maltese, Alfred M. Raccuia, Irena DeMario, Christopher Silvestri.

NY Onicron – St. Joseph's College

Chapter President– Christine Vaccaro, 35 Current Members, 22 New Members

Other fall 2006 officers: Jaclyn Pirrotta, Vice–President; Adrienne Eterno, Secretary; Alicia Gervasi, Treasurer; Elana Epstein, Cor. Sec.

Meetings were held once a month. Students from the seminar class presented their mathematical findings on research they conducted throughout the semester.

New Initiates – Frank Amitrano, Paul Andrejkovics, Christine Bennett, Michele Bramanti, Andrea Chibbaro, Jessica D'Amato, Amanda Drevis, Adrienne Eterno, Matthew Furlani, Alicia Gervasi, Jenna Haines, Nicole Namann, Stanley Hanscom, Matthew Kofsky, Samantha Leibowitz, Jaclyn Pirrotta, Jaclyn Risch, Laura Seidler, Joel Sutherland, Jennifer Wesnofske, Edward D'Azzo-Caisser, Brittany Guardino.

OH Epsilon – Marietta College

Chapter President – Phil DeOrsey, 20 Current Members, 0 New Members

Other fall 2006 officers: Matthew Hunnefeld, Vice-President; Dr. John C. Tynan, Corresponding Secretary.

OH Gamma– Baldwin-Wallace College

Chapter President – Kathleen Turk, 28 Current Members, 20 New Members

Other fall 2006 officers: Gretchen Waugaman, Vice-President; Andrew Miskimen, Secretary; Megan Saad, Treasurer; Dr. David Calvis, Corresponding Secretary.

OK Alpha – Northeastern State University

Chapter President– Lindsey Box, 60 Current Members, 8 New Members

Other fall 2006 officers: Bobbie Back, Vice–President; Seana Smith, Secretary; Jeff Smith, Treasurer; Dr. Joan E. Bell, Corresponding Secretary.

Our fall initiation brought eight new members into our chapter. Our speaker this semester was Dr. Wendell Wyatt, Northeastern State Univ. His presentation, "Geometry in Chinese Architecture," included slides of the buildings and landscaping from a recent trip to China. At one of our meetings we sponsored a Sudoku puzzle contest. Winner was our president, Lindsey Box. We again participated in the annual NSU Halloween carnival with our "KME Pumpkin Patch" activity. The children fished for pumpkins with meter stick fishing poles. Our chapter also participated in the Redmen Rally, a recruitment day for area high school students. We ended the semester with a Christmas party for KME members, math majors, and faculty. The pizza, made by our department chair, Dr. Darryl Linde, was incredible! Special guests at the party were Mr. & Mrs. Maurice Turney. He has been a member of our Oklahoma Alpha chapter since 1945!

New initiates: Phillip D. Howell, Evan Linde, Dustin Little, Felicia Lotchleas, Rebecca Stockstill, Catherine Swanson, Carol Swigert, Moria Yancy.

OK Gamma – Southwest Oklahoma State University

Bill Sticka, Corresponding Secretary.

New Initates – Crystal Clay, Laura Feeley, Anh Tong, Joe Wilson.

PA Beta – LaSalle University

Chapter President – Brian Story, 4 Current Members, 0 New Members

Other fall 2006 officers: Melissa Meyer, Vice-President; Jeremiah Noll, Treasurer; Dr. Anne E. Edlin, Corresponding Secretary.

In conjunction with the MAA student chapter we had a Bowling for Primes evening.

PA Eta – Grove City College

Dale L. McIntyre; Corresponding Secretary.

New initiates – Susan Allgaier, Joshua Inks, Erin Lukasiewicz, Jennifer Nuber, Joshua Rupert, Dustin Kifer, Chad Morley, Justin Peachy, Matthew Sensinger, Jason Simon, Matt Ziders, Sarah Smith, Louise Balwit, Casey Clements, Timothy Hopper, Samantha Gathers, Jennifer Howell, Zachary Kaskan, Jana Kucharik, Andrea Langer, Laura Lunz, Kriatin McCune, Bryan Schwab, Rachel Scott.

PA Epsilon – Kutztown University

Randy S. Schaeffer, Corresponding Secretary.

New Initiates – Demi Heimbach, Rachael Kanusky, Christopher Kavcak, Keith Monihen, Sara Otis, Caitlin Sublette, Abby Bloss, Jenna Dicarlo, Jonathan Dimino, Melissa Ebling, Samantha Fichthorn, Meghan Ghaffari, Mallary Kamen, Christina Klucharich, Jessica Kiscadden, Shaunna Knepp, Angela Lengel, Amy Miller, Swapna Mudigonda, Denise Noll, Jessica Paulas, Tracey Rickert, Robin Lusch, Ruth Melenda, David Rieksts, Charles Swartz VI, Tara Smith, Stanley Walerski, Kerry Wells, Jennifer Wiand, Christy Williams.

PA Iota– Shippensburg University

John Cooper, Corresponding Secretary.

New Initiates – Melinda Meisel, Robin Wolfe, Michelle Baker, Jeff Becker, Fred Donelson, Kaitlin Erb, Bryan Weaver.

PA Lambda – Bloomsburg University of Pennsylvina

Chapter President – Justin Wright, 30 Current Members, 8 New Members
Other fall 2006 officers: April Stepanski, Vice-President; Andrew Walter, Secretary; Anup Sharma, Treasurer; Dr. Elizabeth Mauch, Corresponding Secretary.

New Initiates – Nicole Andriano, Jennifer Blose, Anne Cassel, Corey Dufrene, Larry Kretzing, Christen McDermott, Mark Wilson, Steve Withers.

PA Mu – Saint Francis University

Katherine S. Remillard, Corresponding Secretary.

New Initiates – Jason Burkett, Diane Conrad, Denis Eradiri, Heather Rust, Michael Sharbaugh, Timothy Gaborek, David Kirby, Michael Layton, Joseph Rosmus, Kelleen Skoner, Kelly Slingwine, Kaitlyn Snyder.

PA Nu – Ursinus College

Jeffrey Neslen, Corresponding Secretary.

New Initiates – Ashley Potter, Sara McNally, Dana Bryson, Sylvania Tang, Jason Minutoli, Lauren Rees, J. Bailey Turner.

PA Sigma– Lycoming College

Chapter President – Jessica E. Gough, 11 Current Members, 0 New Members
Other fall 2006 officers: Amanda L. Borden, Vice-President; Elizabeth M. Sullivan, Secretary; Dung A. Tran, Treasurer; Dr Santu de Silva, Corresponding Secretary.

The KME seal was mounted in Welch Honors Hall in time for the Spring Induction. No meetings were held during Fall 2006.

SC Epsilon – Francis Marion University

Chapter President – James Michael McLellan, 4 New Members

Other fall 2006 officers: Tiffany K. Vereen, Vice-President; Jennifer Amy Driggers, Secretary; Matthew Steven Donaldson, Treasurer; Damon Scott, Corresponding Secretary.

New Initiates – Matthew Steven Donaldson, Jennifer Amy Driggers, James Michael McLellan, Tiffany K. Vereen.

TN Gamma – Union University

Chapter President– Kendal Hershberger, 16 Current Members, 0 New Members

Other fall 2006 officers: Joshua Shrewsberry, Vice–President; Matthew Dawson, Secretary; Matthew Dawson, Treasurer; David Moses, Webmaster; Bryan Dawson, Corresponding Secretary.

The TN Gamma chapter sponsored two events this semester. The first was a cookout September 25 at the residence of professor Hail; the Great Dawsoni provided the entertainment with a mathematical trick. The second event was a Christmas party December 7 at the home of professor Lunsford. We held our traditional white elephant gift exchange, and for the second consecutive year a computer (old, but working!) was among the gifts. The party also featured a viewing of the video "The Great Pi/e Debate."

TX Iota – McMurry University

Dr. Kelly L. McCoun, Corresponding Secretary.

New initiates – Lynn Blair, Chris Cumby, Lee Kim, Rosa Ledezma, Tyler McCracken, Juliana Meadows, Nicole Tunmire, David Upshaw, Tammy Werner, Lindsey Raff.

VA Delta – Marymount University

Dr. Elsa Schaefer, Corresponding Secretary.

New Initiates – Katie Armentrout, Jennifer Kikta Marshall, Maureen B. Smith, Emily Parent, Justin Domes

WI Gamma – University of Wisconsin-Eau Claire

Dr. Marc Goulet, Corresponding Secretary.

New Initiates – Riley Abing, Jileen Arendt, Sarah Barlow, Brandon Barrette, Sarah Bianchet, Hallie Kohl, Stacy Kouba, Hiep Cong Nguyen, Derek Olson, Amanda Funk, Brent Haffenbredl, Amy Raplinger, Lori Scardino, Ellen Shafer, Victoria Udalova, Eric Weber, Mitchell Phillipson, Corey Hilber, Elizabeth Wilson, Andrew Yost, Dr. Simei Tong, Ryan Goodrich.

WV Alpha – Bethany College

Dr. Mary Ellen Komorowski, Corresponding Secretary.

New Initiates – William R. Culler, Sabrina Iqbal, David Allen Hayes, Casey Rae Callahan, Jennifer Mae Manor, Mallory Lynn Roadman, Danielle Marie Buck, Bethany McGrail Sloane, Douglas E. Winwood, Ashley Ruth Collett, Brian James Lish, John C. McLane

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Springfield, MO 65802

KME National Website:

http://www.kappamuepsilon.org/

Active Chapters of Kappa Mu Epsilon

Listed by date of installation

Chapter	Location	Installation Date
OK Alpha	Northeastern State University, Tahlequah	18 April 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
MS Beta	Mississippi State University, Mississippi State	14 Dec 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
AL Alpha	Athens State University, Athens	5 March 1935
NM Alpha	University of New Mexico, Albuquerque	28 March 1935
IL Beta	Eastern Illinois University, Charleston	11 April 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 April 1937
OH Alpha	Bowling Green State University, Bowling Green	24 April 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	University of Central Missouri, Warrensburg	10 June 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 June 1941
MI Beta	Central Michigan University, Mount Pleasant	25 April 1942
NJ Beta	Montclair State University, Upper Montclair	21 April 1944
IL Delta	University of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 March 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 June 1947
CO Alpha	Colorado State University, Fort Collins	16 May 1948
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 April 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	University of Nebraska—Kearney, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960
OH Epsilon	Marietta College, Marietta	29 Oct 1960

MO Zeta	University of Missouri—Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood University, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 April 1965
AL Epsilon	Huntingdon College, Montgomery	15 April 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
AR Alpha	Arkansas State University, State University	21 May 1965
TN Gamma	Union University, Jackson	24 May 1965
WI Beta	University of Wisconsin—River Falls, River Falls	25 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	McDaniel College, Westminster	30 May 1965
IL Zeta	Domincan University, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Truman State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel University, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 March 1971
KY Alpha	Eastern Kentucky University, Richmond	27 March 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 April 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973
NY Kappa	Pace University, New York	24 April 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State University, Joplin	8 May 1975
GA Alpha	State University of West Georgia, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin—Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sept 1978
IL Theta	Benedictine University, Lisle	18 May 1979
PA Mu	St. Francis University, Loretto	14 Sept 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C.W. Post Campus of Long Island University, Brookville	2 May 1983
MO Kappa	Drury University, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 March 1985
NE Delta	Nebraska Wesleyan University, Lincoln	18 April 1986

TX Iota	McMurry University, Abilene	25 April 1987
PA Nu	Ursinus College, Collegeville	28 April 1987
VA Gamma	Liberty University, Lynchburg	30 April 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 April 1990
CO Delta	Mesa State College, Grand Junction	27 April 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 April 1991
SD Alpha	Northern State University, Aberdeen	3 May 1992
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 March 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993
MS Epsilon	Delta State University, Cleveland	19 Nov 1994
PA Omicron	University of Pittsburgh at Johnstown, Johnstown	10 April 1997
MI Delta	Hillsdale College, Hillsdale	30 April 1997
MI Epsilon	Kettering University, Flint	28 March 1998
KS Zeta	Southwestern College, Winfield	14 April 1998
TN Epsilon	Bethel College, McKenzie	16 April 1998
MO Mu	Harris-Stowe College, St. Louis	25 April 1998
GA Beta	Georgia College and State University, Milledgeville	25 April 1998
AL Eta	University of West Alabama, Livingston	4 May 1998
NY Xi	Buffalo State College, Buffalo	12 May 1998
NC Delta	High Point University, High Point	24 March 1999
PA Pi	Slippery Rock University, Slippery Rock	19 April 1999
TX Lambda	Trinity University, San Antonio	22 November 1999
GA Gamma	Piedmont College, Demorest	7 April 2000
LA Delta	University of Louisiana, Monroe	11 February 2001
GA Delta	Berry College, Mount Berry	21 April 2001
TX Mu	Schreiner University, Kerrville	28 April 2001
NJ Gamma	Monmouth University	21 April 2002
CA Epsilon	California Baptist University, Riverside	21 April 2003
PA Rho	Thiel College, Greenville	13 February 2004
VA Delta	Marymount University, Arlington	26 March 2004
NY Omicron	St. Joseph's College, Patchogue	1 May 2004
IL Iota	Lewis University, Romeoville	26 February 2005
WV Beta	Wheeling Jesuit University, Wheeling	11 March 2005
SC Epsilon	Francis Marion University, Florence	18 March 2005
PA Sigma	Lycoming College, Williamsport	1 April 2005
MO Nu	Columbia College, Columbia	29 April 2005
MD Epsilon	Villa Julie College, Stevenson	3 December 2005
NJ Delta	Centenary College, Hackettstown	1 December 2006
NY Pi	Mount Saint Mary College, Newburgh	20 March 2007