The Problem Corner

Edited by Pat Costello and Kenneth M. Wilke

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following problems should be submitted on separate sheets before January 1, 2007. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Spring, 2007 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859)622-3051)

CONTINUING PROBLEMS 585, 587, 589

Problem 585. *(Corrected) Proposed by José Luis Diaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Suppose that the roots $z_1, z_2, ..., z_n$ of

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z^1 + a_0 = 0$$

are in arithmetic progression with difference d. Prove that

$$d^2 = \frac{12\left[(n-1)a_{n-1}^2 - 2na_{n-2}\right]}{n^2(n^2-1)}.$$

Problem 587. *Proposed by José Luis Diaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Show that if A, B, C are the angles of a triangle, and a, b, c its sides, then

$$\prod_{\text{cyclic}} \sin^{1/3}(A-B) \leq \sum_{\text{cyclic}} \frac{\left(a^2+b^2\right)\sin(A-B)}{3ab}.$$

Problem 589. *Proposed by the editor.*

Find a, b, c, d, and e so that the number

$$a8b2cd7e3$$

is divisible by both 73 and 137, where a, b, c, d, and e are distinct integers chosen from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and $a > 0$.

NEW PROBLEMS 597-603

Problem 597. *Proposed by Bangteng Xu, Eastern Kentucky University, Richmond KY.*

Determine the following limit.

$$\lim_{n\to\infty} \frac{1 \cdot n + 3\,(n-1) + 5\,(n-2) + \cdots + (2n-3) \cdot 1}{n^3}$$

Problem 598. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA.*

Let C be a unit circle centered at the point $(3, 4)$. Let $O = (0, 0)$, and let $A = (1, 0)$. Let P be a variable point on C, and let $a = PA$ and $b = PO$. Find a non-constant polynomial $f(x, y)$ such that

$$f(a, b) = 0$$

for all points P on C.

Problem 599. *Proposed by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.*

Primes of the form $3n^2 + 3n + 1$ are called *Cuban primes*. Find necessary and sufficient conditions for $3n^2 + 3n + 1$ to be divisible by 7.

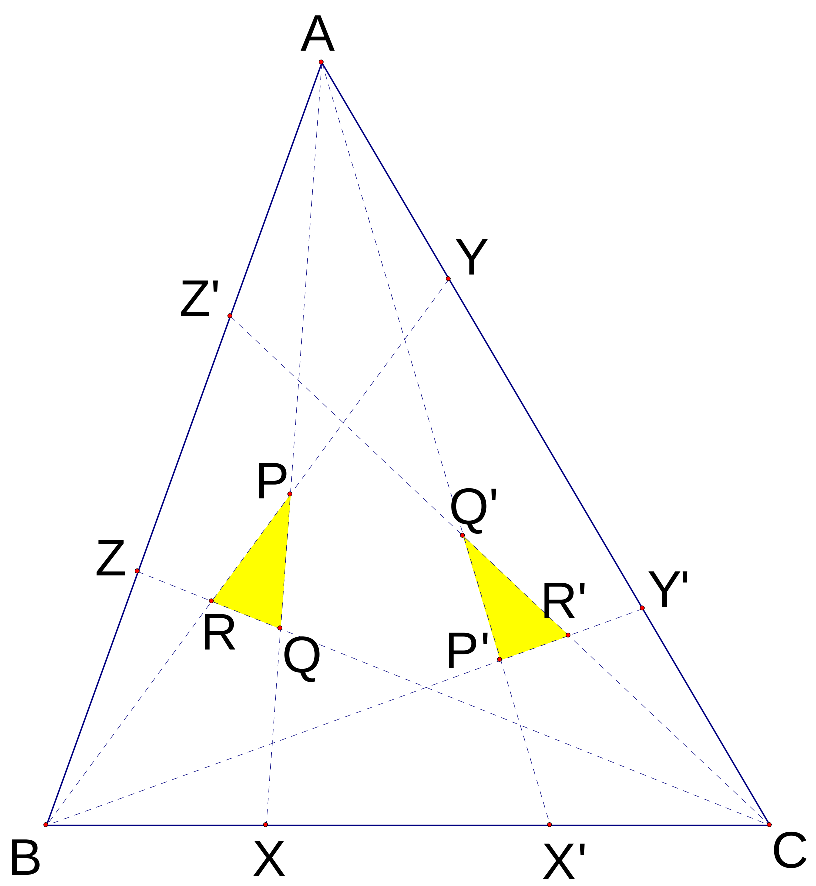
Problem 600. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA.*

In ΔABC, let X, Y, and Z be points on sides BC, CA, and AB, respectively. Let

$$x = \frac{BX}{XC},\ y = \frac{CY}{YA},\ \text{and}\ z = \frac{AZ}{ZB}.$$

The lines AX, BY, CZ bound a central triangle PQR. Let X', Y', and Z' be the reflections of X, Y, and Z, respectively, about the midpoints of the sides of the triangle upon which they reside. These give rise to a central triangle $P'Q'R'$. Prove that the area of ΔPQR is equal to the area of $\Delta P'Q'R'$ if and only if either

$$x = y \text{ or } y = z \text{ or } z = x.$$

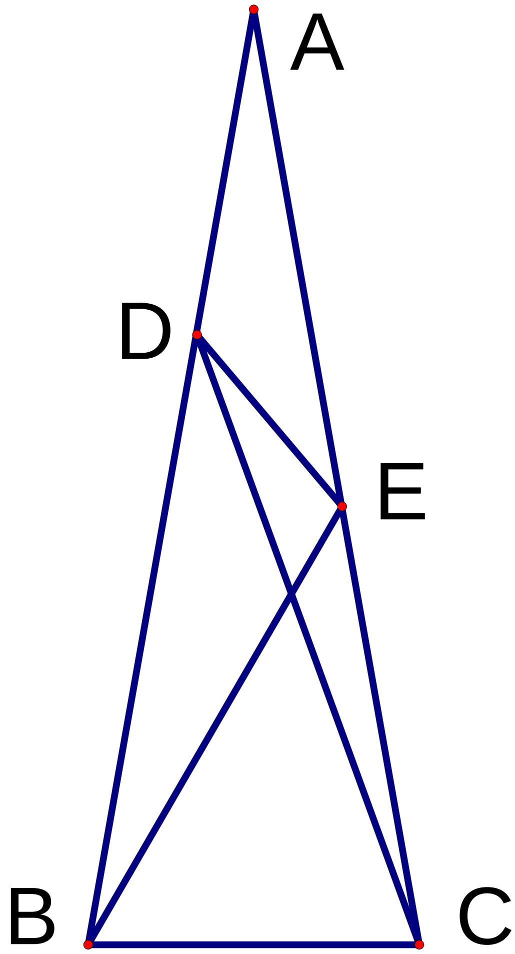


Problem 601. *Proposed by Johannas Winterink, Albuquerque, NM.*

You are given the following information about the drawn triangle:

- Point A, D, and B are collinear;
- Points A, E, and C are collinear;
- $\angle DAE = 20°$, $\angle ADE = 130°$, $\angle AEB = 140°$, $\angle ADC = 150°$.

Prove that $AB = AC$.



Problem 602. *Proposed by the editor.*

Consider the sequence of polynomials defined recursively by

$$\begin{cases} p_1(x) = (x-2)^2 \\ p_2(x) = [p_1(x) - 2]^2 \\ \vdots \\ p_n(x) = [p_{n-1}(x) - 2]^2 \end{cases}.$$

If

$$p_n(x) = x^{2n} + a_{2n-1}x^{2n-1} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_1x + 4,$$

find closed formulas for the coefficients a_{2n-1}, a_{2n-2}, a_2, and a_1.

Problem 603. *Proposed by the editor.*

Consider the following variant on Pascal's triangle. Start with the top two rows the same as in Pascal's triangle. For the remaining rows, put 1 at each end. For each interior entry, add the two diagonal values above the position plus the value in the row above which is between the two summed values. This means an interior entry is a sum of values in the equilateral triangle above the position. Rows 0, 1, 2, 3, and 4 of the triangle are:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 3 & & 1 & & \\ & 1 & & 5 & & 5 & & 1 & \\ 1 & & 7 & & 13 & & 7 & & 1 \end{array}.$$

One notable fact about this triangle is that all entries in the triangle are odd. Another fact is that the second diagonal is the set of odd numbers. Find a closed formula for the sum of the entries across the m^{th} row.

Please help your editor by submitting problem proposals.