The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before August 1, 2010. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall 2010 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859)622-3051)

NEW PROBLEMS 649-658

Problem 649. *Proposed by Tuan Le, Fairmont High School, Anaheim, CA.*

Suppose x, y, z are positive real numbers such that $xyz \geq 10 + 6\sqrt{3}$. Prove that

$$\frac{y}{x+y^3+z^2} + \frac{z}{x^2+y+z^3} + \frac{x}{x^3+y^2+z} \leq \frac{1}{2}$$

Problem 650. *Proposed by Tuan Le, Fairmont High School, Anaheim, CA.*

Suppose a, b, c are positive real numbers. Prove that

$$\frac{a^2b+b^2c+c^2a}{3(a^3+b^3+c^3)} + \frac{a^2+b^2+c^2}{ab+bc+ac)} \geq \frac{4}{3}$$

Problem 651. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Find all triplets (x, y, z) of real numbers for which

$$\sqrt[4x-y^2-2]{\sqrt[4y-z^2-2]{\sqrt[4z-x^2-2]{\sqrt{41x+43y+44z}}}}$$

is a positive integer and determine the values.

Problem 652. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let *a, b, c* be the lengths of the sides of triangle *ABC*. Prove that

$$\left(\frac{\sqrt{bc}}{a}\right)\sin A+\left(\frac{\sqrt{ca}}{b}\right)\sin B+\left(\frac{\sqrt{ab}}{c}\right)\sin C \le \frac{3\sqrt{3}}{2}$$

Problem 653. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN.*

Find the general solution of the recurrence relation $\sum_{k=0}^{2010}(2010)^{2011-k}x_{n-k}=0, \quad n\ge 2010.$

Problem 654. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN.*

Find all real roots of the exponential equation

$$e^{9x} - 2e^{8x} + e^{7x} - 29e^{6x} + 87e^{5x} - 87e^{4x} + 29e^{3x} - e^{2x} + 2e^{x} - 1 = 0.$$

Problem 655. *Proposed by Ovidiu Furdui, Campia Turzh, Cluj, Romania.*

Let *f(n)* be the function defined by *f(n)* = $\ln 2/2^{k-1}$ if $2^{k-1} \le n < 2^k$. Prove that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-f(n)\right) = \gamma$$ where γ denotes the Euler Mascheroni constant.

Problem 656. *Proposed by Ovidiu Furdui, Campia Turzh, Cluj, Romania*.

Let $k \ge 2$ be a natural number. Find the sum $\sum_{n_1,\ldots,n_k\ge 1}(-1)^{n_1+\ldots+n_k}(\zeta(n_1+\ldots+n_k)-1)$ where ζ denotes the Riemann zeta function.

Problem 657. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*
Let *a, b, c* be positive real numbers. Prove that

$$\frac{5c^2+11ab}{(a+b)^2}+\frac{5a^2+11bc}{(b+c)^2}+\frac{5b^2+11ca}{(c+a)^2} \ge 12$$

Problem 658. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let *a, b, c* be the sides, m_a, m_b, m_c, the medians, h_a, h_b, h_c the heights, l_a, l_b, l_c the bisectors and R the circumradius (radius of the circle inside which the triangle can be inscribed) of triangle *ABC*.

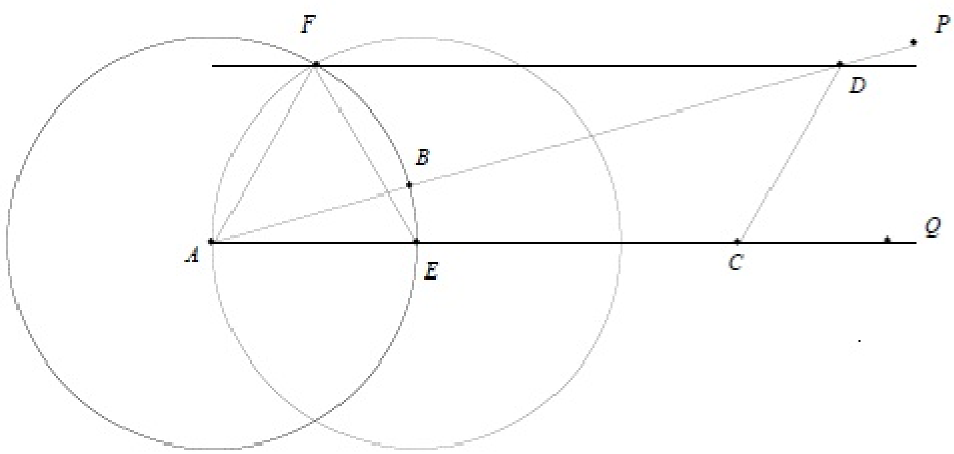
Prove that $$\frac{l_a^2}{h_a}\sqrt{\frac{m_a^2-h_a l_a}{l_a^2-h_a^2}}+\frac{l_b^2}{h_b}\sqrt{\frac{m_b^2-h_b l_b}{l_b^2-h_b^2}}+\frac{l_c^2}{h_c}\sqrt{\frac{m_c^2-h_c l_c}{l_c^2-h_c^2}}\le 6R$$

SOLUTIONS TO PROBLEMS 632-640

Problem 632. *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, IL.*

Let two rays meet at point A and let P be a point on one ray and Q a point on the other ray. Let B be a point between A and P. Suppose the angle measure of $\angle$PAQ is less than $60°$. Show how to construct, with only compass and straightedge, points D on AP and C on AQ such that CD = AB and DC makes an angle of $60°$ with AQ.

Solution *by the proposers.*
Draw a circle with center *A*, radius *AB* to determine a point *E* on *AQ* with *AB* = *AE*. Draw a circle with center *E*, radius *AB* to determine a point *F* such that *AFE* is an equilateral triangle (with *F* on the same side of *AQ* as *B*). Draw *FD* parallel to *AQ* with *D* on *AP* and *DC* parallel to *AF* with *C* on *AQ*. Since *AFDC* is a parallelogram, it is straightforward to show that *CD* = *AB* and that *CD* makes an angle of $60°$ with *AQ*.
The following diagram shows the construction.



Also solved by the Dead Poet Society at Berry College, Mount Berry, GA.

Problem 633. *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, IL.*

The integers beginning with 2008 and without spaces between them are written down.

20082009201020112012 2013...

Then commas are placed to form an infinite sequence of 5-digit arrangements.

20082,00920,10201,12012,2013...

Prove or disprove: Every 5-digit arrangement appears infinitely many times in this sequence.

Solution *by the proposers.*

Let *xyztu* be a 5-digit arrangement. Since all of the numbers are written down, eventually the original list of numbers will contain *xyztuAxyztuBxyztuCxyztuDxyztu* where *A*, *B*, *C*, and *D* are some arbitrary integers. In fact, the list will contain an infinite number of such 29-digit arrangements. When the commas are placed, the 29-digit arrangement will be partitioned into five 5-digit arrangements, plus some digits left over. By considering the various possibilities for the location of the comma in the first 5-digit arrangement, *xyztu*, we see that the list contains one of the following:

,xyztu,Axyzt,uBxyz,tuCxy,ztuDx,yztu
x,yztuA,xyztu,Bxyzt,uCxyz,tuDxy,ztu
xy,ztuAx,yztuB,xyztu,Cxyzt,uDxyz,tu
xyz,tuAxy,ztuBx,yztuC,xyztu,Dxyzt,u
xyzt,uAxyz,tuBxy,ztuCx,yztuD,xyztu

In particular, the list contains the 5-digit sequence *xyztu* an infinite number of times.

Problem 634. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH.*

Find the limit $$\lim_{n\to\infty} \frac{1}{n}\left(\frac{n}{\frac{1}{2}+\frac{2}{3}+\ldots+\frac{n}{n+1}}\right)^n$$

Solution *by the proposer.*
The limit equals $e^{\gamma-1}$. We have
$$\frac{1}{2}+\frac{2}{3}+\ldots+\frac{n}{n+1} = n-\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+1}\right) = n + 1 - \gamma_{n+1} - \ln(n+1)$$
where $\gamma_{n+1} = 1 + ½ + \ldots + 1/(n+1) - \ln(n+1)$. The original terms above become
$$x_n = \frac{1}{n}\left(\frac{1}{1-\frac{\gamma_{n+1}+\ln(n+1)-1}{n}}\right)^n = \frac{1}{n}\left(\frac{1}{1-a_n}\right)^n$$
where $a_n = \dfrac{\gamma_{n+1}+\ln(n+1)-1}{n}$. We note that $a_n = O(\,(\ln n)/n\,)$. Thus
$$\begin{aligned}\ln x_n &= -n\ln(1-a_n) - \ln n = n\,(\,a_n + a_n^2/2 + \ldots\,) - \ln n\\ &= n\,(\,a_n + O(\,(\,(\ln n)/n\,)^2\,) - \ln n\\ &= n\,a_n - \ln n + n\,O(\,(\,(\ln n)/n\,)^2\,)\\ &= \gamma_{n+1} - 1 + \ln(\,(n+1)/n\,) + n\,O(\,(\,(\ln n)/n\,)^2\,)\end{aligned}$$
Letting $n \to \infty$, we get the desired result since $n\,O(\,(\,(\ln n)/n\,)^2\,) \to 0$.

Problem 635. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH.*

Let $k \geq 2$ be a natural number. Find the sum $\sum_{n_1,\ldots,n_k\geq 1} (\zeta(n_1+\ldots+n_k)-1)$ where ζ denotes the Riemann zeta function.

Solution *by the proposer.*
The series equals $\zeta(k)$. We have
$$\sum_{n_1,\ldots,n_k\geq 1} (\zeta(n_1+\ldots+n_k)-1) = \sum_{n_1,\ldots,n_k\geq 1}\left(\sum_{p=2}^{\infty}\frac{1}{p^{n_1+\ldots+n_k}}\right) = \sum_{p=2}^{\infty}\left(\sum_{n_1=1}^{\infty}\frac{1}{p^{n_1}}\right)\cdots\left(\sum_{n_k=1}^{\infty}\frac{1}{p^{n_k}}\right)$$

Since these sums are geometric series with sum $1/(p-1)$, we get that

$$\sum_{n_1,\ldots,n_k\geq 1} (\zeta(n_1+\ldots+n_k)-1) = \sum_{p=2}^{\infty}\frac{1}{(p-1)^k} = \zeta(k).$$

Problem 636. *Proposed by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.*

Let p be a fixed prime. Find the dimensions of all rectangles with integral side lengths and whose areas are numerically equal to p times their semiperimeters.

Solution *by the proposers.*
Let x and y be the dimensions of the rectangle. One needs to solve the Diophantine equation $xy = p(x + y)$. Notice that $x \neq p$. For if it were, then $py = p(x + y)$ and so $x = 0$. Solving the equation for y gives $y = p + p^2/(x-p)$. Since y must be a positive integer, $x - p$ must be a divisor of p^2. Since p is a prime, $x - p = -p^2, -p, -1, 1, p$, or p^2. Since x must be a positive integer, $x = p - 1,\ p + 1, 2p$, or $p + p^2$. So $y = p - p^2,\ p^2 + p, 2p$, or $p + 1$, respectively. Since the first of the y values will not be positive, the possible pairs (x, y) are $(p+1, p^2+p)$, $(2p,2p)$, or $(p^2+p, p+1)$.

Problem 637. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let $x, y, z \in [1, +\infty)$. Prove that $\dfrac{x}{x^2+yz} + \dfrac{y}{y^2+xz} + \dfrac{z}{z^2+xy} \leq \dfrac{3}{2}$

Solution *by the Dead Poet Society at Berry College.*
Pick $x, y, z \in [1, \infty)$. Then we have $\dfrac{x}{x^2+yz} \leq \dfrac{1}{2}$ if and only if $\dfrac{x^2+yz}{x} \geq 2$. Notice that since $y, z \geq 1$ we have $\dfrac{x^2+yz}{x} \geq \dfrac{x^2+1}{x}$. Now notice that $\dfrac{x^2+1}{x} \geq 2$ iff $x^2 + 1 \geq 2x$ iff $x^2 - 2x + 1 \geq 0$ iff $(x - 1)^2 \geq 0$ which of course is true. Hence by transitivity we have that $\dfrac{x}{x^2+yz} \leq \dfrac{1}{2}$. But the three terms on the left hand side of the desired inequality are symmetric in x, y, and z. Therefore each term is less than or equal to ½. Therefore,

$$\frac{x}{x^2+yz} + \frac{y}{y^2+xz} + \frac{z}{z^2+xy} \leq \frac{3}{2}$$

Also solved by the proposer and the Oklahoma Alpha Chapter, Northeastern State University, Tahlequah, OK.

Problem 638. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

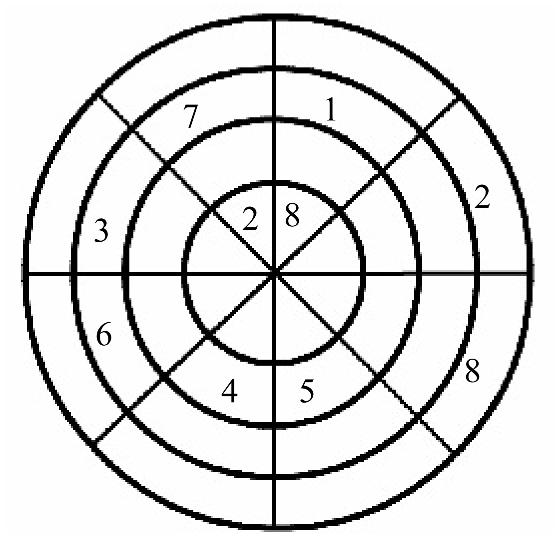
Let a be a positive integer. Find the least common multiple of the numbers $A = a^{n}(a+1)^{n+1} + a$ and $B = a^{n+1}(a+1)^{n} + a - 1$ where n is any natural number.

Solution *by the proposer.*

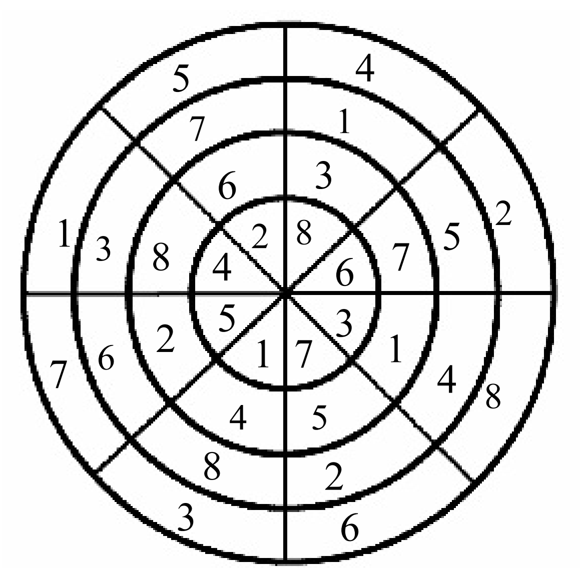
Let us denote the GCD and LCM of A and B by (A,B) and $[A,B]$, respectively. If $d = (A, B)$, then $d \mid a^{n}(a+1)^{n+1} + a$ and $d \mid a^{n+1}(a+1)^{n} + a - 1$. Hence we have that $d \mid a[a^{n}(a+1)^{n+1} + a\ ]$ and $d \mid (a+1)[a^{n+1}(a+1)^{n} + a - 1\ ]$, or equivalently $d \mid a^{n+1}(a+1)^{n+1} + a^{2}$ and $d \mid a^{n+1}(a+1)^{n+1} + a^{2} - 1$. Thus d divides their difference. That is, $d \mid 1$. Therefore, $(A, B) = 1$ and taking into account that $(A, B)[A,B] = AB$, as is well-known, we get $[A,B] = [\ a^{n}(a+1)^{n+1} + a\ ][\ a^{n+1}(a+1)^{n} + a - 1\ ] = a^{2n+1}(a+1)^{2n+1} + a^{n}(a+1)^{n}\ (2a^{2} - 1) + a\ (a - 1)$ and we are done.

Problem 639. *Proposed by Peter M. Higgins and Caroline Higgins (authors of the book Circular Sudoku), Essex University, England.*

The following is a Circular Sudoku puzzle. Each of the numbers 1-8 must appear once in every ring and once in every pair of touching slices. Fill in the missing values of the puzzle.
ring and once in every pair of touching slices. Fill in the missing values of the puzzle.



Solution *by the proposer.*



Also solved by the Dead Poet Society at Berry College, Mount Berry, GA; Sister Marcella Wallowicz, Holy Family University, Philadelphia, PA; Oklahoma Alpha Chapter, Northeastern State University, Tahlequah, OK; Joshua Liberman, Craig Edwards at Central Missouri State University, Warrensburg, MO; Timothy Groh, John Gruenewald, Hunter Hayes, Paige Hebard, Michael Henry, Lucas Hoots, Lisa Notier, Morgan Smith, Brian Sneed at Centre College, Danville, KY; Eddie Bailey, Amanda Bowling, Donald Brewer, Jeremy Britt, Matt Columbus,

Jenny Fischesser, Sarah Hale, Hunter Nuckols, Joe Snyder at Eastern Kentucky University, Richmond, KY

Problem 640. *Proposed by the editor.*

The sequence 19, 199, 1999, ... starts off with three primes, however, most of the numbers in the sequence are composites and there are lots of divisors of the numbers in the sequence. Prove the following items:

a) The prime 19 actually divides infinitely many of the numbers in the sequence.
b) The composite number 551 divides infinitely many of the numbers in the sequence.
c) The composite number 323 does not divide any of the numbers in the sequence.

Solution *by the proposer.* The numbers in the sequence are of the form $2*10^n - 1$ where the exponent n creates the value having $n+1$ digits.

a) All the numbers in the sequence having $18m + 2$ digits will be divisible by 19. To see this,

$10^{18m+1} = 10\,(10^{18})^m \equiv 10*1^m \pmod{19}$ by Fermat's Little Theorem.

So $10^{18m+1} \equiv 10 \pmod{19}$ for all positive integers m. This means that

$2*10^{18m+1} - 1 \equiv 0 \pmod{19}$ or 19 divides all values in the sequence having $18m+2$ digits.

Note further that $1 \pmod{18}$ is the only exponent on 10 in the last congruence that gives 0.

b) All the numbers in the sequence having $252m + 74$ digits will be divisible by 551. To see this, we start by observing that $10^{28m+17} = 10^{17}\,(10^{28})^m \equiv 15*(10^{28})^m \pmod{29} \equiv 15*1^m \pmod{29}$ by Fermat's Little Theorem. So 29 divides all values in the sequence having $28m+18$ digits. Since $551 = 19*29$, we need to see if any of the values from part a) will have $28n+18$ digits for some n. This will happen if the system of linear congruences

$m \equiv 1 \pmod{18}$ and $m \equiv 17 \pmod{28}$

has any solutions. This system has solution $m \equiv 73 \pmod{252}$. Therefore 551 divides all values in the sequence having $252n + 74$ digits.

c) Since $323 = 17*19$, we would need a subsequence of the values in part a) that are divisible by 17 in order for the values to be divisible by 323. When we consider powers of 10 (mod 17), we see that

$10^{16m+6} = 10^6\,(10^{16})^m \equiv 10^6 *1^m \pmod{17}$ by Fermat's Little Theorem.

So $2*10^{16m+6} - 1 \equiv 0 \pmod{17}$ or 17 divides all values in the sequence having $16m+7$ digits. Moreover, these are the only values in the sequence that are divisible by 17. Since these values have an odd number of digits and the values divisible by 19 have an even number of digits, there are no values divisible by both 17 and 19.