The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before January 1, 2010. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Spring 2010 issue of The Pentagon. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu)

NEW PROBLEMS 641-648

Problem 641. *Proposed by Lisa Kay, Eastern Kentucky University, Richmond, KY.*

Suppose that there are five students enrolled in a chemistry class. They will have to complete five lab assignments. For each lab assignment, four of the students will work in two pairs while one student works independently. Each student will work independently for exactly one of the five labs. Each student will work with each of the other four students exactly once. How many different lab schedules are possible?

Problem 642. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let a, b, c be the lengths of the sides of a triangle ABC with heights h_a, h_b, and h_c, respectively. Prove that

$$\prod_{\text{cyclic}} \left(\frac{h_a}{h_b + h_c}\right)^{1/3} \leq \frac{1}{6}\left(\frac{a+b+c}{\sqrt[3]{abc}}\right).$$

Problem 643. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

The equation $x^3 - 2x^2 - x + 1 = 0$ has three real roots $a > b > c$. Find the value of $ab^2 + bc^2 + ca^2$.

Problem 644. *Proposed by Andrew Cusumano, Great Neck, NY.*

Find two primes whose reciprocals repeat after exactly 7 decimal places.

Problem 645. *Proposed by Ben Thurston, Florida Southern College, Lakeland, FL.*

What is the expected number of rolls of a fair die required to have all six faces come up at least once?

Problem 646. *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, Illinois.*

Suppose that n is an odd integer. Show that

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{n-1} - \frac{1}{n} > \frac{1}{4} - \frac{1}{2\,(n+1)}.$$

Problem 647. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let a, b, and c be the sides, and m_a, m_b, and m_c the medians of a triangle ABC. Prove or disprove that

$$m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2 \geq \frac{9}{4}\left(\frac{a^4 bc\cos A}{b^2+c^2} + \frac{ab^4 c\cos B}{c^2+a^2} + \frac{abc^4\cos C}{a^2+b^2}\right).$$

Problem 648. *Proposed Ovidiu Furdui, University of Toledo, Toledo, OH.*

Let $k > 1$ be a real number. Find the value of $\int_0^1 \left\{\frac{1}{\sqrt[k]{x}}\right\} dx$, where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a. [For example, $\{1.9\} = 0.9$.]

SOLUTIONS TO PROBLEMS 624-631

Problem 624. *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, Illinois.*

Given a tetrahedron, prove that two triangles can be formed such that the lengths of the six triangle sides equal the lengths of the six edges of the tetrahedron. Prove that the converse is not true.

Solution *by the proposers.*

Suppose the six edges of the tetrahedron cannot be arranged to form two triangles. Let the vertices be labeled so that AB is the longest edge of the tetrahedron. If

$$AB < AC + AD,$$

then AB, AC, and AD can be arranged into one triangle and $\triangle BCD$ is a triangular face of the tetrahedron so we would have two triangles. This means that

$$AB \geq AC + AD.$$

Similarly,

$$AB \geq BC + BD.$$

Therefore,

$$2AB \geq AC + AD + BC + BD.$$

However, $\triangle ABC$ is a face of the tetrahedron, so

$$AB < AC + BC.$$

Since $\triangle ABD$ is also a face,

$$AB < AD + BD.$$

Adding these two inequalities gives

$$2AB < AC + BC + AD + BD.$$

This contradicts the previous inequality. Hence the six edges can be arranged to form two triangles.

To prove that the converse is not true, consider the two triangles one being equilateral whose sides have length 1 and the other having sides of length 100, 102, 104. With these lengths, we can form only one triangle with the length 104 edge as one side of the triangle. But each edge of a tetrahedron is an edge of two of the triangles formed by the faces. Therefore, it is not possible to arrange these six edges into a tetrahedron.

Problem 625. *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, Illinois.*

All of the integers from 1 through 999999 are written in a row. All of the zeros are erased. Each of the remaining digits is separately inverted and the sum, S, is computed. Let T be the sum of the reciprocals of the digits 1 through 9. Show that S/T is an integer and find it.

Solution *by the proposers.*

Consider the one million six-digit sequences, $xyztuv$, where x, y, z, t, u, and v are between 0 and 9, inclusive. In each of these one million sequences, let each digit be replaced by that digit plus 1 modulo 10 (so that 9 is replaced by 0). The resulting list of sequences is the same, except for order, as the initial list. Thus each digit occurs the same number of times among the one million six-digit sequences. As there are 6,000,000 digits, each of the ten digits occurs 600,000 times. Now suppose all of the zeroes are erased from among the one million six-digit sequences. The digits that remain will be the same as those that would be left if all integers from 1 through 999,999 were written in a row and then all of the zeroes were erased. In each case, each of the digits 1 through 9 occurs 600,000 times. Hence $S = (600,000)T$ and

$$S/T = 600,000.$$

Solution *by Samantha Corvino (student), Slippery Rock University, Slippery Rock, PA.*

Let S_n be the sum of the reciprocals of the nonzero digits of the positive integers less than 10n. Clearly $S_1 = T$. For two-digit numbers, each nonzero digit appears ten times in the tens column and 9 times in the ones column. Thus,

$$S_2 = 10T + 9S_1 + S_1 = 10T + 10S_1.$$

Similarly,

$$S_3 = 100T + 9S_2 + S_2 = 10^2T + 10S_2,$$

and in general

$$S_n = 10^{n-1}T + 10S_{n-1}.$$

One can easily verify that the solution to this difference equation has the form

$$S_n = n10^{n-1}T,$$

from which it can be deduced that $S = S_6 = 600,000T$ and

$$S/T = 600,000.$$

Also solved by Carl Libis, University of Rhode Island, Kingston, RI, Taylor Franzman (student), California State University-Fresno, Fresno, CA, Erik Murphy (student), Waynesburg University, Waynesburg, PA, Parker Richey (student), Northeastern Oklahoma State University, Tahlequah, OK.

Problem 626. *Proposed by David Rose, Florida Southern College, Lakeland, FL.*

Two values are randomly selected from the uniform distribution on the interval $(0, L)$. They create three subintervals of the interval $[0, L]$. What is the probability that the lengths of the three subintervals are the lengths of the sides of some triangle?

Solution *by Chip Curtis, Missouri Southern State University, Joplin, MO.*

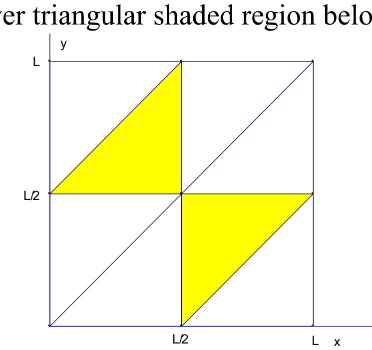
Regard the two randomly-selected values as coordinates of an ordered pair (x, y) in the xy-plane. The feasible region is the square with vertices $(0, 0)$, $(L, 0)$, $(0, L)$, and (L, L), together with its interior.

Case 1 Suppose $0 < x \le y < L$. In this case, the three segment lengths are x, $y - x$, and $L - y$. These lengths will be side lengths for a triangle if and only if they satisfy all three inequalities required by the triangle inequality:

$$\begin{aligned} L - y &< x + (y - x) = y \Longleftrightarrow y > L/2 \\ y - x &< L - y + x \Longleftrightarrow y < L/2 + x \\ x &< L - y + y - x = L - x \Longleftrightarrow x < L/2 \end{aligned}$$

The solution to this system of inequalities corresponds to the upper triangular shaded region below.

Case 2 Suppose $0 < y < x < L$. The segment lengths are y, $x - y$, and $L - x$, with those corresponding to sides of a triangle when the x and y values are in the lower triangular shaded region below.



Since the total shaded area represents $\frac{1}{4}$ of the total area of the square, the probability that the lengths constructed are the sides of a triangle is $\frac{1}{4}$.

Also solved by Lisa Kay, Eastern Kentucky University, Richmond, KY, Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, and the proposer.

Problem 627. *Proposed by Ken Dutch, Eastern Kentucky University, Richmond, KY.*

Suppose that the artist Krypto wants to form several rows of blocks 10 feet wide. He only wants to use two types of blocks - one type is one foot wide and the other is two feet wide. He wants to form a row for every possible pattern of blocks (order matters). How many rows will he have to make? How many of each type of block will he have to use?

Solution *by Alycia Butchelli and Karissa King (students), Slippery Rock University, Slippery Rock, PA.*

There are only six basic combinations of the blocks as described below. Each must be permuted. Given that there are repeated elements in each combination, we obtain:

	m	d	P
□ □ □ □ □ □ □ □ □ □	10	0	$\frac{10!}{10!} = 1$
□ □ □ □ □ □ □ □ ▭	8	1	$\frac{9!}{8!1!} = 9$
□ □ □ □ □ □ ▭ ▭	6	2	$\frac{8!}{6!2!} = 28$
□ □ □ □ ▭ ▭ ▭	4	3	$\frac{7!}{4!3!} = 35$
□ □ ▭ ▭ ▭ ▭	2	4	$\frac{6!}{2!4!} = 15$
▭ ▭ ▭ ▭ ▭	0	5	$\frac{5!}{5!} = 1$

,

where m is the number of monominoes, d is the number of dominoes, and P is the number of permutations.

There are 89 permutations, so there are 89 rows needed. To find the number of each individual block type being used, multiply the number of arrangements for each combination by the number of each type of block in the combination and add. The number of monominoes used is 420. The number of dominoes used is 235.

Also solved by Carl Libis, University of Rhode Island, Kingston, RI and the proposer.

Problem 628. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let F_n be the n^{th} Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that

$$\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} F_k \tanh F_k\right)^2 + \left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} F_k \operatorname{sech} F_k\right)^2 \leq F_n F_{n+1}.$$

Solution *by the proposer.*

Applying the Cauchy-Schwarz inequality to the vectors $\overrightarrow{u} = (F_1, F_2, ..., F_n)$ and $\overrightarrow{v} = (\tanh F_1, \tanh F_2, ..., \tanh F_n)$ and taking into account that

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1},$$

as can be easily proved, we get

$$\begin{aligned} \left(\sum_{k=1}^{n} F_k \tanh F_k\right)^2 &\leq \left(F_1^2 + F_2^2 + \cdots + F_n^2\right)\left(\sum_{k=1}^{n} \tanh^2 F_k\right) \\ &= F_n F_{n+1}\left(\sum_{k=1}^{n} \tanh^2 F_k\right). \end{aligned}$$

Applying the Cauchy-Schwarz inequality to the vectors $\overrightarrow{u} = (F_1, F_2, ..., F_n)$ and $\overrightarrow{v} = (\operatorname{sech} F_1, \operatorname{sech} F_2, ... \operatorname{sech} F_n)$, we have

$$\begin{aligned} \left(\sum_{k=1}^{n} F_k \operatorname{sech} F_k\right)^2 &\leq \left(F_1^2 + F_2^2 + \cdots + F_n^2\right)\left(\sum_{k=1}^{n} \operatorname{sech}^2 F_k\right) \\ &= F_n F_{n+1}\left(\sum_{k=1}^{n} \operatorname{sech}^2 F_k\right). \end{aligned}$$

Since $\tanh^2 F_k + \operatorname{sech}^2 F_k = 1$ (in general, $\tanh^2 x + \operatorname{sech}^2 x = 1$), we obtain, after adding the previous expressions,

$$\left(\sum_{k=1}^{n} F_k \tanh F_k\right)^2 + \left(\sum_{k=1}^{n} F_k \operatorname{sech} F_k\right)^2 \leq n F_n F_{n+1}$$

from which the desired conclusion follows. Equality holds when $n = 1$.

Problem 629. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let a, b, c be real numbers, with $a, b, c \geq 1$. Prove that

$$\frac{a^{1/a}}{b^{1/b}+c^{1/c}}+\frac{b^{1/b}}{a^{1/a}+c^{1/c}}+\frac{c^{1/c}}{b^{1/b}+a^{1/a}}<2.$$

Solution *by the proposer.*

First, we will see that with segments of lengths $a^{1/a}$, $b^{1/b}$, $c^{1/c}$, it is always possible to build a triangle. In fact, we have $[a] \leq a < a+1$. Applying Bernoulli's inequality yields $2^a \geq 2^{[a]} = (1+1)^{[a]} \geq 1+[a] > a \geq 1$ from which we get $2 > a^{1/a} \geq 1$. Likewise $2 > b^{1/b} \geq 1$ and $2 > c^{1/c} \geq 1$ and

$$\begin{aligned} a^{1/a}+b^{1/b} &\geq 1+1=2>c^{1/c} \\ b^{1/b}+c^{1/c} &\geq 1+1=2>a^{1/c} \\ c^{1/c}+a^{1/a} &\geq 1+1=2>b^{1/b}. \end{aligned}$$

Therefore it is always possible to build a triangle with the lengths $a^{1/a}$, $b^{1/b}$, $c^{1/c}$. Now we have

$$\begin{aligned} a^{1/a}+b^{1/b} &> \left(a^{1/a}+b^{1/b}+c^{1/c}\right)/2 \\ b^{1/b}+c^{1/c} &\geq \left(a^{1/a}+b^{1/b}+c^{1/c}\right)/2 \\ c^{1/c}+a^{1/a} &\geq \left(a^{1/a}+b^{1/b}+c^{1/c}\right)/2. \end{aligned}$$

Inverting the preceding inequalities, and multiplying by $c^{1/c}$, $a^{1/a}$, $b^{1/b}$, respectively, we obtain

$$\begin{aligned} \frac{c^{1/c}}{a^{1/a}+b^{1/b}} &< \frac{2c^{1/c}}{a^{1/a}+b^{1/b}+c^{1/c}} \\ \frac{a^{1/a}}{c^{1/c}+b^{1/b}} &< \frac{2a^{1/a}}{a^{1/a}+b^{1/b}+c^{1/c}} \\ \frac{b^{1/b}}{a^{1/a}+b^{1/b}} &< \frac{2b^{1/b}}{a^{1/a}+b^{1/b}+c^{1/c}}. \end{aligned}$$

Adding these inequalities and rearranging terms gives the desired conclusion.

Problem 630. *Proposed by the editor.*

Suppose that $\log_x y + \log_y x$ is a positive integer. Prove that $(\log_x y)^n + \left(\log_y x\right)^n$ is an integer for all positive integers n.

Solution *by Daniel Parkes and Kevin Rose (students), California State University-Fresno, Fresno, CA.*

Let $s = \log_y x$ and $t = \log_x y$. Then $y^s = x$ and $x^t = y$. By substitution, $\left(x^t\right)^s = y^s = x$, so $x^{st} = x^1$. Equating exponents, $st = 1$, so that s and t are inverses. We proceed by induction. The smallest case is $n = 1$ which is the same as the original assumption. Assume that $s^k + t^k$ is an integer for all k with $1 \leq k \leq n$. Consider the case of $s^{n+1} + t^{n+1}$. We break into two cases:

Case 1 $n + 1$ even, so $n + 1 = 2u$ for some integer u. Then
$(s+t)^{n+1} = s^{n+1} + a_1 s^n t + a_2 s^{n-1} t^2 + \cdots + a_u s^u t^u + \cdots + a_1 s t^n + t^{n+1}$,
with the a_i being binomial coefficients. Since $st = 1$, we have

$$\begin{aligned}
(s+t)^{n+1} &= s^{n+1} + a_1 s^{n-1} + a_2 s^{n-3} + \cdots + a_u + \cdots \\
&\quad + a_1 t^{n-1} + t^{n+1} \\
&= s^{n+1} + a_1\left(s^{n-1} + t^{n-1}\right) + a_2\left(s^{n-3} + t^{n-3}\right) + \cdots \\
&\quad + a_u + t^{n+1}.
\end{aligned}$$

By the induction hypothesis, $s^{n-1} + t^{n-1}$, $s^{n-3} + t^{n-3}$, ... are all integers. Since all of the binomial coefficients are integers and $(s+t)^{n+1}$ is an integer, $s^{n+1} + t^{n+1}$ is an integer.

Case 2 $n + 1$ odd, so $n = 2u$ for some integer u. Then

$$\begin{aligned}
(s+t)^{n+1} &= s^{n+1} + a_1 s^n t + a_2 s^{n-1} t^2 + \cdots + a_u s^{u+1} t^u + a_u s^u t^{u+1} \\
&\quad + \cdots + a_1 s t^n + t^{n+1} \\
&= s^{n+1} + a_1 s^{n-1} + a_2 s^{n-3} + \cdots + a_u s + a_u t \\
&\quad + \cdots + a_1 t^{n-1} + t^{n+1} \\
&= s^{n+1} + a_1\left(s^{n-1} + t^{n-1}\right) + a_2\left(s^{n-3} + t^{n-3}\right) \\
&\quad + \cdots + a_u (s+t) + t^{n+1}.
\end{aligned}$$

As in case 1, all of the middle terms on the right are integers, and the left side is an integer, so $s^{n+1} + t^{n+1}$ is an integer.

Thus by mathematical induction, $(\log_x y)^n + \left(\log_y x\right)^n$ is an integer for all positive integers n.

Also solved by Joan Bell, Northeastern Oklahoma State University, Tahlequah, OK, Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO, and the proposer.

Problem 631. *Proposed by the editor.*

The Columbus State University Problem of the Week for March 10, 2008 asked for the three smallest positive integers that could not be written as the difference of two positive prime numbers. These turn out to be primes. Prove that there are infinitely many positive primes that cannot be written as the difference of two positive prime integers. Also prove that there are infinitely many pairs of positive integers $(n, n+2)$ that cannot be written as the difference of two positive primes.

Solution *by the proposer.*

Let p be a prime with $p \equiv 3 \pmod{10}$. Then p is an odd prime. If p can be written as the difference of two positive primes, say $p = q - r$, then $q > p$ and so q is odd. Since p and q are odd, r would need to be even. The only even prime is 2 which forces $q = p + 2 \equiv 5 \pmod{10}$ which is impossible if q is prime. Hence p cannot be written as the difference of two primes. By Dirichlet's Theorem on Primes in Arithmetic Progressions we know there are infinitely many primes congruent to 3 $(\mathrm{mod}\, 10)$ and none of these can be written as the difference of two primes.

For the second part of the problem, let n be a prime with $n \equiv 3 \pmod{70}$. So n is odd. If n can be written as the difference of two positive primes, say $n = q - r$, then q would be odd, $r = 2$, and $q = n + 2 \equiv 5 \pmod{70}$ which is impossible for a prime q. Hence n cannot be written as the difference of two primes. In addition, if $n + 2$ can be written as the difference of two primes, say $q - r$, then q would be odd, $r = 2$, and $q = (n + 2) + 2 \equiv 7 \pmod{70}$ which is also impossible. So neither n nor $n + 2$ can be written as the difference of two primes. Since 3 is relatively prime to 70, Dirichlet's Theorem on Primes in Arithmetic Progressions guarantees that there are infinitely many primes $n \equiv 3 \pmod{70}$. So there are infinitely many pairs of positive integers $(n, n+2)$ that cannot be written as the difference of two positive primes.