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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

The General Dice Problem

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The general dice problem may be stated thus: What is the probability of throwing the sum s in a single throw with n dice each of which has k faces? This problem is clearly a generalization of the "crap shooter's" problem, what is the probability of throwing the sum 8 (or any other possible sum) in a single throw with two ordinary dice. The solution of the general problem can be obtained by a generalization of the solution of this particular one. The possibility of a die with k faces will be discussed later.

We start then with the sum of 8 and two dice. Throwing a die is equivalent to taking the exponent of a term chosen at random from the six terms of the polynomial $x^1 + x^2 + x^3 + x^4 + x^5 + x^6$. Throwing two dice is equivalent to taking the exponents of the two terms chosen at random one from each of the factors of the product

(1)
$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$$

The number of ways of throwing the sum 8 is then the coefficient of x^8 in the product (1). This follows because, to obtain the product (1), it is necessary to multiply each term in the first parenthesis by each term in the second parenthesis. The number of ways that this can be done to obtain x^8, say x^3 by x^5, x^5 by x^3, x^2 by x^6, etc., will be the coefficient of x^8 in the product. The coefficient of x^8 in the expansion of $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^2$ is then the number of equally likely ways in which we can throw the sum 8 with two dice. The total number of equally likely ways of throwing two dice is 6^2. Hence our required probability is the above coefficient divided by 6^2.

The generalization of this now becomes obvious. If we had n dice instead of two, we would need to consider the expansion of

(2)
$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^n.$$

If we want the probability of the sum s, the number of equally likely ways of obtaining it is the coefficient of x^s in the expansion of (2). Finally, if our dice have k faces, we would need the coefficient of x^s in the expansion of

(3)
$$(x^1 + x^2 + x^3 + \cdots + x^k)^n$$

for the equally likely ways of obtaining the sum s. This number divided by k^n, the number of possible ways of throwing n dice with k faces each, would give the required probability. To obtain the coefficient of x^s, we write (3) as

$$x^n(1 + x + x^2 + \cdots + x^{k-1})^n = x^n[(1 - x^k)/(1 - x)]^n$$
$$= x^n(1 - x^k)^n(1 - x)^{-n},$$

where $[(1 - x^k)/(1 - x)]$ is the sum of the geometric series

$$1 + x + x^2 + \cdots + x^{k-1}.$$

Then
$$(1 - x)^{-n} = 1 + (-n)(-x) + [(-n)(-n-1)/2!](-x)^2 + [(-n)(-n-1)(-n-2)/3!](-x)^3 + \cdots,$$

or

(4)
$$(1 - x)^{-n} = 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \binom{n+2}{3}x^3 + \cdots,$$

where $\binom{m}{r}$ represents $m!/[r!(m-r)!]$. Also,

$$(5) \quad x^n(1 - x^k)^n = x^n - \binom{n}{1}x^{n+k} + \binom{n}{2}x^{n+2k} - \binom{n}{3}x^{n+3k} + \cdots + (-1)^n x^{n+nk}.$$

By multiplying the right members of (4) and (5), we obtain the expansion of (3). To obtain the terms in x^s in this expansion, we must multiply the successive terms in (5) by the following terms in order from (4):

$$\binom{s-1}{s-n}x^{s-n}, \binom{s-k-1}{s-n-k}x^{s-n-k}, \binom{s-2k-1}{s-n-2k}x^{s-n-2k}, \cdots.$$

This sequence continues as long as the powers of x are positive. Collecting the terms in x^s, we obtain the coefficient of x^s as

$$\binom{s-1}{s-n} - \binom{n}{1}\binom{s-k-1}{s-n-k} + \binom{n}{2}\binom{s-2k-1}{s-n-2k} - \binom{n}{3}\binom{s-3k-1}{s-n-3k} + \cdots.$$

By a simple transformation this becomes

$$\binom{s-1}{n-1} - \binom{n}{1}\binom{s-1-k}{n-1} + \binom{n}{2}\binom{s-1-2k}{n-1} - \binom{n}{3}\binom{s-1-3k}{n-1} + \cdots.$$

This series terminates when any symbol takes an impossible form.

For example, let $s = 8$, $n = 2$, and $k = 6$, which is the "crap shooter's" problem mentioned previously. Then we have from (6): $\binom{7}{1} - \binom{2}{1}\binom{1}{1} = 7 - 2 = 5$. Our probability is 5/36. For a second example, let $s = 10$, $n = 3$, and $k = 6$. We have from (6): $\binom{9}{2} - \binom{3}{1}\binom{3}{2} = 36 - 3 \cdot 3 = 27$. Our probability is $27/6^3 = 27/216$.

Now consider the possible values of k, the number of faces on our dice. Since a die must have an equal chance of coming to rest on any face, it would seem necessary that all faces be congruent and that faces be surrounded in exactly the same way by polyhedral angles. The five regular convex polyhedrons, the rhombic dodecahedron, and the rhombic triacontahedron fulfill these requirements, to mention a few. These would give k the values 4, 6, 8, 12, 20, 30. Incidentally, all of them except the regular tetrahedron have their faces parallel in pairs so that there would be no question as to which face was uppermost. In the case of the regular tetrahedron, we might consider the number thrown to be the number marked on the face upon which the die comes to rest.

"There is an astonishing imagination, even in the science of mathematics • • •. We repeat, there was far more imagination in the head of Archimedes than in that of Homer."

—VOLTAIRE

A New Approach To Ceva's Theorem

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In studying the properties of a triangle, sooner or later one is concerned with a set of three lines, one through each vertex and intersecting the opposite side. There are many sets of lines with these properties, altitudes, bisectors, medians, etc. In this paper a proof of Ceva's Theorem is given and by assigning special values to parameters the altitudes, bisectors, and medians become special cases.

A family of three lines passing through the vertices of a triangle and intersecting the opposite sides of the triangle has many representations. The representation that is used here is based on the parametric equations of the sides of the triangle. A parameter t_a, t_b, t_c is assigned to each side a, b, c of the triangle ABC. By assigning to these parameters certain values, the equations and points of intersection of the medians, bisectors, and altitudes are obtained.

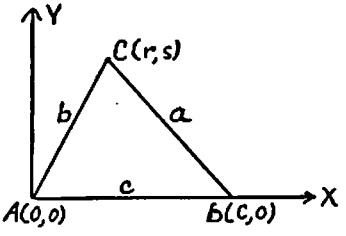

Figure 1

Given a triangle ABC with sides a, b, and c, a rectangular coordinate system is set up by letting the vertex A be the origin, the side c lying on the positive half of the x-axis. The coordinates of the vertices are A $(0, 0)$, B $(c, 0)$, and C (r, s). The coordinates r and s may be expressed in terms of the lengths of the sides of the triangle. By the law of cosines,

$$\cos A = (b^2 + c^2 - a^2)/(2bc) = r/b.$$

$$\text{Thus } r = (b^2 + c^2 - a^2)/(2c).$$

The formulas for the area of a triangle give

$A = \frac{1}{2}sc = \sqrt{l(l-a)(l-b)(l-c)}$; $l = \frac{1}{2}(a+b+c)$;
thus, $s = 2A/c = 2\sqrt{l(l-a)(l-b)(l-c)}/c$.

The parametric equations of the three sides of the triangle are:

$$\begin{aligned} a:\quad & x = c + t_a(r-c), \\ & y = t_a s; \\ b:\quad & x = r(1-t_b), \\ & y = s(1-t_b); \\ c:\quad & x = t_c c, \\ & y = 0; \end{aligned}$$

where t_i is the parameter on the side i of the triangle. When t_i takes on values between 0 and 1 the point (x,y) moves along the side i from one vertex to the other in a counterclockwise direction around the triangle.

The lines through the vertices A, B, C and intersecting the opposite sides a, b, c will be denoted by Aa, Bb, Cc. It is convenient to write the equations of these lines in parametric form with parameter w and as functions of x and functions of y. As w takes on values from 0 to 1 the point (x,y) moves along Aa (or Bb or Cc) from the vertex to the opposite side of the triangle.

$$\begin{aligned} Aa:\quad & x = w[c + t_a(r-c)], \\ & y = wt_a s; \\ Bb:\quad & x = c + w[(1-t_b)r - c], \\ & y = w(1-t_b)s; \\ Cc:\quad & x = r + w(t_c c - r), \\ & y = s(1-w). \end{aligned}$$

Eliminating the parameter w and obtaining x as a function of y and y as a function of x gives:

$$\begin{aligned} Aa:\quad & x = y[c + t_a(r-c)]/(t_a s) \\ & y = (t_a s x)/[c + t_a(r-c)] \\ Bb:\quad & x = y[(1-t_b)r - c]/[(1-t_b)s] + c \\ & y = x[(1-t_b)s]/[(1-t_b)r - c] \\ & \quad - [(1-t_b)cs]/[(1-t_b)r - c] \end{aligned}$$

$$Cc: \quad x = y(r - t_c c)/s + t_c c$$
$$y = x[s/(r - t_c c)] - (t_c cs)/(r - t_c c)$$

These three lines will generally have three points of intersection. Solving these equations simultaneously the three points of intersection are:

$$Aa \cap Bb: \quad x = (1 - t_b)[(1 - t_a)c + t_a r]/(1 - t_b + t_a t_b)$$
$$y = [t_a(1 - t_b)s]/(1 - t_b + t_a t_b)$$
$$Bb \cap Cc: \quad x = [(1 - t_b)(1 - t_c)r - t_b t_c c]/(1 - t_c + t_b t_c)$$
$$y = [(1 - t_b)(1 - t_c)s]/(1 - t_c + t_b t_c)$$
$$Cc \cap Aa: \quad x = t_c[(1 - t_a)c + t_a r]/(1 - t_a + t_c t_a)$$
$$y = (t_a t_c s)/(1 - t_a + t_c t_a)$$

If the three lines Aa, Bb, Cc intersect in only one point then the three points of intersection obtained above must be the same point. Equating the corresponding coordinates of the above points and simplifying gives the following equation:

$$1) \quad 2t_a t_b t_c - (t_a t_b + t_b t_c + t_c t_a) + (t_a + t_b + t_c) - 1 = 0,$$
$$\text{or} \quad t_a t_b t_c = (1 - t_a)(1 - t_b)(1 - t_c).$$

Theorem: A necessary and sufficient condition that the lines Aa, Bb, Cc intersect in a point is that the parameters t_a, t_b, t_c satisfy the above (Equation 1).

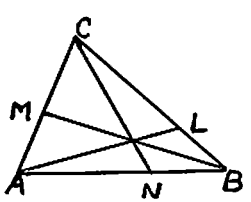

Figure 2

This theorem is equivalent to Ceva's Theorem and its converse. Ceva's Theorem states that if three concurrent lines are drawn from the vertices of a triangle ABC to meet the opposite sides at L, M, N respectively then $(BL/LC)(CM/MA)(AN/NB) = 1$.

Let t_a be the parameter that determines the point L on the side a of the triangle. Then the ratio BL/LC of the segments of the side of the

triangle can be written as $t_a/(1 - t_a)$ and similarly $CM/MA = t_b/(1 - t_b)$ and $AN/NB = t_c/(1 - t_c)$. Substituting in the conclusion of Ceva's Theorem gives:

$$(BL/LC)(CM/MA)(AN/NB) = [t_a/(1 - t_a)][t_b/(1 - t_b)][t_c/(1 - t_c)] = 1.$$

Simplifying the equation gives

$$t_a t_b t_c = 1 - t_a - t_b - t_c + t_a t_b + t_b t_c + t_c t_a - t_a t_b t_c.$$

The medians are obtained when $t_a = t_b = t_c = ½$. Substituting these values of t_a, t_b, t_c in the equations of the sides of the triangles gives the midpoints of the sides a, b, c as $[(c + r)/2, s/2]$, $(r/2, s/2)$, $(c/2, 0)$. The equations of the medians of the triangle are obtained by substituting these values of t_a, t_b, t_c in the equations Aa, Bb, Cc. This gives

$$M_a: \quad y = sx/(2r).$$
$$M_b: \quad y = s(x - c)/(r - 2c).$$
$$M_c: \quad y = s(2x - c)/(2r - c).$$

Since Equation 1) is satisfied by $t_a = t_b = t_c = ½$ the medians meet in a point. Substituting in $Aa \cap Bb$ gives $[s/3, (r + c)/3]$ as the point of intersection. Substituting these values for x and y in the parametric form of equation Aa and solving for w gives $w = 2/3$; *i.e.*, the medians meet in a point 2/3 of the distance from the vertex to the opposite side.

The bisectors of the interior angles of the triangle are obtained when

$$t_a = c/(c + b); t_b = a/(a + c); t_c = b/(b + a).$$

The value of t_a was obtained by solving for t_a in the equations of the bisector of $\angle A$ and the line Aa. The values of t_b and t_c are obtained by cyclic permutations of the letters a, b, and c.

The bisectors of the angles divide the opposite sides of the triangles in the ratio of $t_i/(1 - t_i)$. For example,

$$t_a/(1 - t_a) = [c/(c + b)]/[1 - c/(c + b)] = c/b.$$

This is the theorem: The bisectors of an interior angle of a triangle divides the opposite side into segments having the same ratio as the other two sides of the triangle.

The above values for t_a, t_b, t_c satisfy Equation 1) so the bisectors meet in a point. From $Aa \cap Bb$ this point has the x-coordinate $c(b - r)/(a + b + c)$ and the y-coordinate $cs/(a + b + c)$. This point is the center of the incircle whose radius is $cs/(a + b + c)$.

Bisectors of the exterior angles are obtained from

$$t_a = c/(c - b); t_b = a/(a - c); t_c = b/(b - a).$$

Substitution in the equations Aa, Bb, Cc, gives the bisector at

$$\begin{aligned} A&: \quad y = sx/(b - c) \\ B&: \quad y = s(c - x)/(c - a - r) \\ C&: \quad y = [(b - a)sx - bcs]/(br - ar - bc) \end{aligned}$$

Since the Equation 1) is not satisfied by these values for t_a, t_b, t_c then bisectors of the exterior angles do not meet in a point. However, the centers of the excircles may be quickly obtained by substitutions in $Aa \cap Bb$, $Bb \cap Cc$, and $Cc \cap Aa$.

The parameters for two exterior angle bisectors and the parameter for the interior bisector of the third angle satisfy Equation 1) so they meet in a common point, the center of one of the excircles. The external bisectors divide the opposite sides into segments having the ratio $t_i/(1 - t_i)$. For example:

$$t_c/(1 - t_c) = [b/(b - a)]/[-a/(b - a)] = -b/a.$$

The negative sign indicates that the point of division is an external point of side c. The bisector of an exterior angle of a triangle divides the opposite side externally into segments having the same numerical ratio as the other two sides of the triangle.

The parameters for the altitudes are

$$t_a = (c^2 + a^2 - b^2)/(2a^2); t_b = (a^2 + b^2 - c^2)/(2b^2);$$
$$t_c = (b^2 + c^2 - a^2)/(2c^2).$$

These parameters satisfy Equation 1) and thus the altitudes meet in a point.

"Many small make a great."

—CHAUCER

Five Mutually Tangent Spheres, Tetrahedrons, And Their Related Problems

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1. Introduction. There exist many interesting problems and relationships among spheres and tetrahedrons. Some of the problems have recently been solved while others remain to be solved. This paper will list the general solutions to three problems: that of five mutually tangent spheres, the volume of an irregular tetrahedron, and the radius of a sphere circumscribed about an irregular tetrahedron. The solutions of related problems will also be treated.

The interesting properties of the special tetrahedron formed by lines connecting the centers of four mutually externally tangent spheres will also be given. The fifth and sixth spheres, mutually tangent to the four given spheres, are so related to this tetrahedron that knowing their radii will simplify finding its volume and the radius of its inscribed and circumscribed spheres. Finally, there will follow several applications of the relationships between five mutually tangent spheres.

2. Five Mutually Tangent Spheres. It is possible to have any three spheres tangent to each other as in Figure 1. Let the centers of these three spheres be considered to lie in a horizontal plane. Then it will be possible to place a fourth sphere above the three mutually

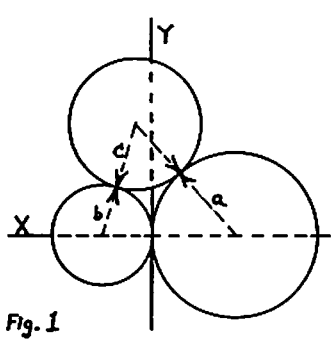

Fig. 1

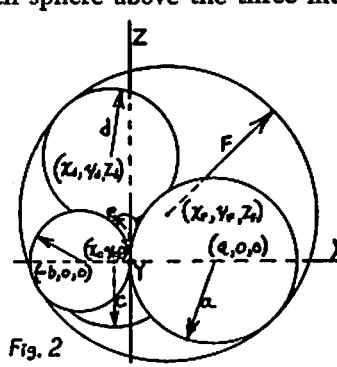

Fig. 2

tangent spheres so that it will be tangent to each of them. The only restriction is that the fourth sphere be large enough so that it does not fall through the space between the three spheres. If the fourth

sphere is small enough so that it does not intersect a plane tangent above to the first three spheres, a fifth sphere can be placed so that it will be tangent to the upper sides of the four given spheres. If the fourth sphere does project above the plane tangent to the first three, then a fifth and largest sphere can enclose the given four spheres so that they will be internally tangent to the largest sphere as in Figure 2. A limiting case exists when the fourth sphere is also tangent to the plane tangent to the first three. Then the fifth sphere becomes a plane surface, *i.e.*, a sphere with an infinite radius.

Therefore, for any four given spheres, there are two other spheres tangent to them, both externally, or one small one externally and one large one to which they are internally tangent.

Let the radii of the four given spheres be a, b, c, and d. Of the two spheres which can be tangent to them let the smaller have a radius of e. Let the larger one to which they could be tangent either externally or internally have a radius of f. The most general expression relating five mutually tangent spheres is[1]

$$(1)\qquad \begin{aligned} a^2b^2c^2d^2 + a^2b^2c^2e^2 + a^2b^2d^2e^2 + a^2c^2d^2e^2 + b^2c^2d^2e^2 \\ - abcde(abc + abd + abe + acd \\ + ace + ade + bcd + bce + bde + cde) = 0. \end{aligned}$$

Equation (1) is a quadratic equation which is symmetrical with respect to all variables.

$$\text{Let}\qquad (2)\qquad \begin{aligned} g &= abc + abd + acd + bcd, \\ k &= ab + ac + ad + bc + bd + cd, \\ n^2 &= a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2, \\ j &= abcd. \end{aligned}$$

The following relationship exists among these new variables:

$$(3)\qquad g^2 = n^2 + 2jk.$$

Writing (1) as a quadratic in e,

$$(4)\qquad \begin{aligned} e^2[(a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2) \\ -abcd(ab + ac + ad + bc + bd + cd)] \\ -e(abcd)(abc + abd + acd + bcd) + a^2b^2c^2d^2 = 0. \end{aligned}$$

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[1] Harvey E. Fiala, "Five Mutually Tangent Spheres." The Pentagon, Spring, 1958.

Rewriting (4) in terms of g, k, n^2 and j by using relationships (2),

$$e^2(n^2 - jk) - ejg + j^2 = 0. \tag{5}$$

and solving for e the following expressions can be obtained:

$$\begin{aligned} e &= [jg \pm \sqrt{(jg)^2 - 4j^2(n^2 - jk)}]/[2(n^2 - jk)] \\ &= j[g \pm \sqrt{3(g^2 - 2n^2)}]/(3n^2 - g^2). \\ &= 2j/[g \mp \sqrt{3(g^2 - 2n^2)}] \end{aligned} \tag{6}$$

Since (1) is symmetrical with respect to all variables, e could be replaced by f. The plus and minus signs in (6) indicate two solutions. It turns out that the solution using the sign resulting in the larger answer is correctly interpreted as the solution for f, while the other solution gives the smaller radius e. However, when the other four mutually tangent spheres are internally tangent to the sphere of radius f (*i.e.*, it encloses all of them), the solution having the larger numerical value will be negative. A negative solution, then, is interpreted as the radius of a sphere that encloses and is tangent to four given spheres. One equation expressing both e and f can then be written

$$e, f = j[g \pm \sqrt{3(g^2 - 2n^2)}]/(3n^2 - g^2). \tag{7}$$

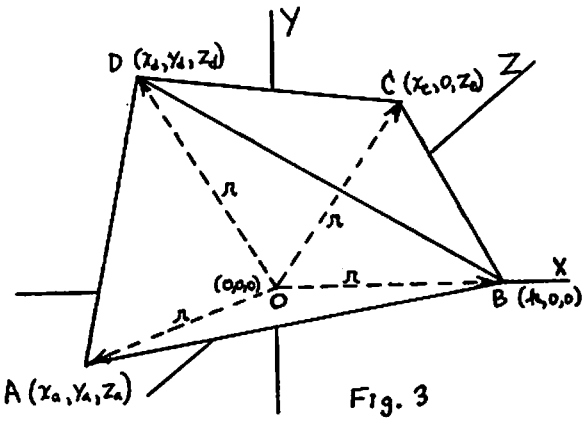

Fig. 3

3. Tetrahedrons. The following ten relations will be necessary

for deriving an expression for the radius of a sphere circumscribed about the irregular tetrahedron $ABCD$.[2] (Refer to Figure 3).

$$(8)\qquad r^2 = \overline{OA}^2 = x_a^2 + y_a^2 + z_a^2.$$

$$(9)\qquad r^2 = \overline{OB}^2 = x_b^2.$$

$$(10)\qquad r^2 = \overline{OC}^2 = x_c^2 + z_c^2.$$

$$(11)\qquad r^2 = \overline{OD}^2 = x_d^2 + y_d^2 + z_d^2.$$

$$(12)\qquad \overline{AB}^2 = (r - x_a)^2 + y_a^2 + z_a^2 = 2r^2 - 2rx_a.$$

$$(13)\qquad \overline{AC}^2 = (x_a - x_c)^2 + y_a^2 + (z_a - z_c)^2 = 2r^2 - 2x_ax_c - 2z_az_c.$$

$$(14)\qquad \overline{AD}^2 = (x_a - x_d)^2 + (y_a - y_d)^2 + (z_a - z_d)^2 = 2r^2 - 2x_ax_d - 2y_ay_d - 2z_az_d.$$

$$(15)\qquad \overline{BC}^2 = (r - x_c)^2 + z_c^2 = 2r^2 - 2rx_c.$$

$$(16)\qquad \overline{BD}^2 = (r - x_d)^2 + y_d^2 + z_d^2 = 2r^2 - 2rx_d.$$

$$(17)\qquad \overline{CD}^2 = (x_c - x_d)^2 + y_d^2 + (z_c - z_d)^2 = 2r^2 - 2x_cx_d - 2z_cz_d.$$

A method of solving these equations for r consists of solving (8) and (11) for the product y_ay_d and substituting this into (14). Then solve (14) for z_a in terms of z_d, solve (17) for z_d in terms of z_c, solve (10) for z_c in terms of x_c and then substitute all into (14) in terms of x's and r. The x's can then be expressed in terms of the six sides AB, AC, AD, BC, BD, and CD and r by solving equations (12), (15), and (16) respectively for x_a, x_c, and x_d. Substitution for all x's into (14) will then leave an equation in terms of only the six sides and r. By cancelling common factors and expanding, all terms in r^4 and r^6 drop out. The resulting expression for r^2 is the following:

$$(18)\qquad \begin{aligned} 4r^2 = {} & [(\overline{AB}^2\overline{CD}^2 + \overline{AC}^2\overline{BD}^2 + \overline{AD}^2\overline{BC}^2)^2 \\ & - 2(\overline{AB}^4\overline{CD}^4 + \overline{AC}^4\overline{BD}^4 + \overline{AD}^4\overline{BC}^4)] \\ & /[\overline{AB}^2\overline{BC}^2(\overline{AD}^2 + \overline{CD}^2 - \overline{AC}^2) + \overline{AB}^2\overline{BD}^2(\overline{AC}^2 \\ & + \overline{CD}^2 - \overline{AD}^2) + \overline{BC}^2\overline{BD}^2(\overline{AC}^2 + \overline{AD}^2 - \overline{CD}^2) \\ & + \overline{AC}^2\overline{AD}^2(\overline{BC}^2 + \overline{BD}^2) + \overline{AC}^2\overline{CD}^2(\overline{AB}^2 + \overline{BD}^2) \\ & + \overline{AD}^2\overline{CD}^2(\overline{AB}^2 + \overline{BC}^2) \end{aligned}$$

[2] The method of solution for the radius of the circumscribed sphere is outlined here because the author believes its solution has never before been published.

$$- \overline{AC}^2\overline{BD}^2(\overline{AC}^2 + \overline{BD}^2) - \overline{AD}^2\overline{BC}^2(\overline{AD}^2 + \overline{BC}^2)$$
$$- \overline{AB}^2\overline{CD}^2(\overline{AB}^2 + \overline{CD}^2) - \overline{AC}^2\overline{AD}^2\overline{CD}^2].$$

The volume V of an irregular tetrahedron can be expressed in terms of the lengths of the six sides of the equation.[3]

$$\begin{aligned}(19)\quad 144V^2 = {} & [(\overline{AB} + \overline{AD} + \overline{BD})(\overline{AB} + \overline{AD} - \overline{BD}) \\ & (\overline{AC} + \overline{BC} - \overline{AB})(\overline{AB} + \overline{BC} - \overline{AC}) \\ & (\overline{AD} + \overline{CD} - \overline{AC})(\overline{AC} + \overline{CD} - \overline{AD})] \\ & - [(\overline{AD})(\overline{AC} + \overline{BC} - \overline{AB})(\overline{AB} + \overline{BC} - \overline{AC}) \\ & + (\overline{AB})(\overline{AD} + \overline{CD} - \overline{AC})(\overline{AC} + \overline{CD} - \overline{AD}) \\ & - (\overline{AC})(\overline{AD} + \overline{BD} - \overline{AB})(\overline{AB} + \overline{BD} - \overline{AD})]^2.\end{aligned}$$

Let A_a be the area of ΔBCD (the face opposite vertex A), A_b the area of ΔACD, etc. Also let h_a be the altitude from the vertex A to the face opposite this vertex, h_b the altitude from vertex B, etc. The altitude from any vertex (say B) to the face opposite this vertex can be expressed by the equation

$$(20)\qquad h_b = 3V/A_b$$

where the volume V can be determined from (19).

The radius R of a sphere inscribed in an irregular tetrahedron is given by (21), where S, defined by (22), is the total surface of the tetrahedron. (See Footnote 2).

$$(21)\qquad R = 3V/S$$

$$(22)\qquad S = A_a + A_b + A_c + A_d$$

Let the dihedral angle between two faces, say ΔDAB and ΔCAB, of a tetrahedron be represented by the symbol $\angle AB$, where $\overline{AB}$ is the edge common to the two faces. The sine of a dihedral angle can be expressed as the ratio of the altitude of the tetrahedron and the altitude of one of its triangular surfaces, where the altitudes are measured from a vertex to the proper base and edge. This results in equation (23) which is an expression for the dihedral angle of a tetrahedron.

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[3] Harvey Fiala, "A Note on Tetrahedrons," Pentagon, Spring, 1956. Equation (19) above is the result of expanding in terms of the six sides the expression for V in the article "A Note on Tetrahedrons." Expressions for the radius of the inscribed sphere and the dihedral angle are also derived there.

$$\angle AB = \sin^{-1}[3V(\overline{AB})/(2A_cA_d)] \tag{23}$$

4. Special Tetrahedron. If a tetrahedron is formed by lines connecting the centers of four mutually tangent spheres, then the following relationships exist:

$$\begin{aligned} \overline{AB} &= a + b & \overline{AC} &= a + c & \overline{AD} &= a + d \\ \overline{BC} &= b + c & \overline{BD} &= b + d & \overline{CD} &= c + d \end{aligned} \tag{24}$$

where $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, and $\overline{CD}$ are the lengths of the six sides and a, b, c, and d are the radii of the four spheres having their centers at the four vertices of the tetrahedron. Not all tetrahedrons can be represented by four radii instead of six sides, but many of the commonly occurring ones are special cases and can be represented in this manner.

For this special tetrahedron it will be shown that the radii e and f of the fifth and sixth tangent spheres can be used to advantage in deriving other formulas for these special tetrahedrons.

Substituting relationships (24) and (2) into the denominator of (18), the denominator reduces to $(g^2 - 2n^2)$. The expression for the radius of the sphere circumscribed about this tetrahedron then becomes

$$\begin{aligned} 4r^2 = [&(\overline{AB}^2\overline{CD}^2 + \overline{AC}^2\overline{BD}^2 + \overline{AD}^2\overline{BC}^2)^2 \\ &- 2(\overline{AB}^4\overline{CD}^4 + \overline{AC}^4\overline{BD}^4 + \overline{AD}^4\overline{BC}^4)]/(g^2 - 2n^2), \end{aligned} \tag{25}$$

where the terms of the numerator are of degree 8 and hence cannot be simplified using relationships (24) and (2).

Substituting relationships (24) and (2) into (19), the expression for the volume of a tetrahedron becomes

$$3V = \sqrt{g^2 - 2n^2}. \tag{26}$$

This expression is considerably shorter and easier to use than is (19). Using (7) and taking the ratio of the difference of f and e to the product of f and e results in another expression for V in terms of the six radii a, b, c, d, e, and f.

$$V = [abcd(f - e)]/3ef\sqrt{3} \tag{27}$$

Substituting for V from (27) into (21), the relationship between the five tangent spheres and the inscribed sphere can be seen.

$$(28)\quad R = 3V/S = \sqrt{g^2 - 2n^2}/S = [abcd(f - e)]/Sef\sqrt{3}$$

Similarly, the expression for the dihedral angle becomes,

$$(29)\quad \angle AB = \sin^{-1}[(a + b)abcd(f - e)/(2\ efA_cA_d\sqrt{3})]$$

It can be seen that there is a very intimate relationship between the properties and the various spheres associated with this special tetrahedron. The expressions for the radius of the inscribed and circumscribed spheres, the volume, and the dihedral angle were all made much simpler and more useable.

5. Applications. Some relationships in two dimensions can easily be derived from the general expressions for radii and volume in three dimensions. From (26) we see that the volume of the tetrahedron formed by lines connecting the centers of four tangent spheres is expressed in terms of the radii of the spheres. If we set this expression for volume equal to zero, we get:

$$g^2 = 2n^2, \text{ or}$$

$$(30)\quad a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2 - 2abcd(ab + ac + ad + bc + bd + cd) = 0$$

This equation expresses the relationship between the radii of four tangent circles. It is similar to (1) except that it has one less variable. That it should represent four tangent circles is only logical since when we set the volume equal to zero, it means that the fourth sphere becomes the smallest and has moved down in between the other three with its center in the plane in which the other three centers lie. Four tangent spheres all having their centers in the same plane determine a degenerate tetrahedron and the intersection of these spheres with the plane is composed of four tangent circles.

Equation (30) can be solved for any radius, yielding the form:

$$(31)\quad d = (abc)/[(ab + ac + bc) \mp 2\sqrt{abc(a + b + c)}]$$

Consider what would happen if we let the denominator of equation (6) approach zero. In this case f has a radius increasing indefinitely and the sphere would approach as a limit a plane surface, tangent to the four given spheres. This would correspond to five tangent spheres, four of which are resting on a plane surface. As the demoninator approaches zero,

$$\text{Limit } f = \sqrt{3j/4k}, \text{ or}$$

$$(32) \quad a^2b^2c + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2 - abcd(ab + ac + ad + bc + bd + cd) = 0$$

It should be noted that these are all special cases of one general formula. All of these special cases are in themselves general expressions for a lower order of events.

The same reasoning can be applied on each of these lower levels. For instance, the denominator of (31) can be set to approach zero to show the fourth circle has an indefinitely increasing radius in this case. This would correspond to four tangent circles, three of which are tangent to the same straight line.

To show an actual computation involving the equation for five tangent spheres, we might ask what is the smallest sphere that will hold three spheres, each with a 2 inch radius, and then what is the next largest size sphere that will also fit into this sphere along with the given three? This problem has a direct application in the design of an inertial guidance system. The large sphere represents the inner spherical gimbal ring while the three smaller spheres represent the stabilizing gyros, one along each of the three axes, while all other smaller components are integrating accelerometers.

The equation for four tangent circles can be used to solve for the large sphere.

$$(33) \quad d = [(2)(2)(2)]/[(4 + 4 + 4) \pm 2\sqrt{8(2 + 2 + 2)}] = -4.31 \text{ and } + .31$$

-4.31 (since it is negative) corresponds to the radius of the outer sphere which contains the other three.

Now substituting radii of 2, 2, 2, and -4.31 into relationships (2) and then into the simplest equation of (6),

$$e = -6.9/(-4.37 \pm 0) = 1.58$$

The results are that a sphere of radius 4.31 inches or diameter of 8.62 inches would hold three 4 inch diameter spheres, and two 3.16 inch diameter spheres.

This article has tried to set forth some basic and some interesting formulas that do not receive treatment in books on solid geometry. The configuration of five tangent spheres has many fascinating properties.

Mechanical Quadrature Formulas

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In the study of the evaluation of a definite integral, assuming the limits are given, two conditions must hold for an exact evaluation. First, the function to be integrated (or integrand) must be known, and second, the function must be integrable. When these conditions are not satisfied, an alternate course of evaluation is pursued, *i.e.*, numerical integration. If we think of integration as the process of finding the area under a curve and the curve represents a function of a single variable, then numerical integration is often referred to as "mechanical quadrature."

Numerical integration is the process of computing the value of a definite integral from a set of numerical values of the integrand for given values of the independent variable. This is necessary in the applied sciences where we may obtain values of the function from observations but may not be capable or desirous of expressing the data by means of a functional expression. The process involves the substitution of an appropriate expression, generally a polynomial, for the integrand. This polynomial approximates the actual function which the empirical data represent. The approximation, according to a theorem stated by Weierstrass in 1885, may be accomplished to any desired degree of accuracy. Briefly the theorem states that for every function $f(x)$ which is continuous in an interval (a, b), it is possible to find a polynomial $P(x)$ such that $|f(x)-P(x)| < \varepsilon$ for every value of x in the interval (a,b), where ε is any preassigned positive value. In many instances the degree of accuracy desired is tempered by the degree of the polynomial with which the applied mathematician or engineer is willing to work.

Once we have obtained a polynomial which represents our integrand, integration term by term may be accomplished and the definite integral evaluated. However, as stated before, since in many cases the polynomial function may not be desired and our only interest lies in the value of the definite integral, the intermediate step of obtaining the polynomial function may be bypassed if quadrature formulas based upon integration of general polynomial approximations may be obtained. We shall now develop these formulas.

Given the set of values (x_0,y_0), (x_1,y_1), $\cdots$ (x_n,y_n) such that the values of x are spaced at equal intervals, or $x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1} = \Delta x = h$, then Newton's Binomial Interpolation Formula represents one form of the desired polynomial approximation to be used as the integrand and has the form

$$\begin{aligned} y = y_0 &+ \Delta y_0(x - x_0)/h + \Delta^2 y_0(x - x_0)(x - x_1)/(2!h^2) \\ &+ \Delta^3 y_0(x - x_0)(x - x_1)(x - x_2)/(3!h^3) + \cdots \\ &+ \Delta^{n-1} y_0(x - x_0)(x - x_1)(x - x_2) \\ &\qquad \cdots (x - x_{n-2})/[(n - 1)!h^{n-1}] \end{aligned}$$

where $\Delta^n y_0 = y_n - \binom{n}{1} y_{n-1} + \binom{n}{2} y_{n-2} - \cdots + (-1)^n y_0$

The derivation of this form may be found in the early chapters of any standard text on the calculus of finite differences.

Integrating Newton's formula over n equal intervals of width $h = \Delta x$, we have

$$\begin{aligned} \int_{x_0}^{x_0 + nh} y\,dx &= \int_{x_0}^{x_0 + nh} [y_0 + \Delta y_0(x - x_0)/h \\ &\qquad + \Delta^2 y_0(x - x_0)(x - x_1)/(2!h^2) + \cdots \\ &\qquad + \Delta^{n-1} y_0(x - x_0) \cdots (x - x_{n-2})/(n - 1)!h^{n-1}]dx \\ &= [y_0 x + \Delta y_0(x^2/2 - x_0 x)/h + \Delta^2 y_0(x^3/3 \\ &\qquad - (x_0 + x_1)x^2/2 + x_0 x_1 x)/(2!h^2) + \cdots]_{x_0}^{x_0 + nh} \\ &= h[ny_0 + (n^2/2)\,\Delta y_0 + (n^3/3 - n^2/2)\Delta^2 y_0/2 \\ &\qquad + (n^4/4 - n^3 + n^2)\Delta^3 y_0/3 + \cdots] \end{aligned}$$

This represents the basic quadrature formula over n equal intervals from which we may obtain specific formulas over any given number of points. For instance, if we let $n = 1$, thus assuming only two points (x_0,y_0) and (x_1,y_1) given and considering only the first difference, we have

$$\int_{x_0}^{x_0+h} y\,dx = \int_{x_0}^{x_1} y\,dx = h[y_0 + \tfrac{1}{2}\Delta y_0]$$
$$= h[y_0 + \tfrac{1}{2}(y_1 - y_0)]$$
$$= (h/2)(y_1 + y_0),$$

which we recognize as the trapezoidal rule. Of course the rule here applies over one interval but may be extended in the usual manner which we shall illustrate in the next derivation.

Had we let $n = 2$, the formula becomes (neglecting all differences above the second since we assume only three values of y),

$$\int_{x_0}^{x_0+2h} y\,dx = \int_{x_0}^{x_2} y\,dx = h[2y_0 + 2\Delta y_0 + (8/3 - 2)\Delta^2 y_0/2]$$
$$= h[2y_0 + 2(y_1 - y_0) + (1/3)(y_2 - 2y_1 + y_0)]$$
$$= (h/3)[y_2 + 4y_1 + y_0].$$

This is Simpson's Rule for approximate integration. Extending Simpson's Rule,

if $$\int_{x_0}^{x_0+2h} y\,dx = (h/3)(y_2 + 4y_1 + y_0),$$

then $$\int_{x_0+2h}^{x_0+4h} y\,dx = (h/3)(y_4 + 4y_3 + y_2),$$

$$\int_{x_0+4h}^{x_0+6h} y\,dx = (h/3)(y_6 + 4y_5 + y_4),$$

. . . . . .

$$\int_{x_0 + (2n-2)h}^{x_0 + 2nh} ydx = (h/3)[y_{2n} + 4y_{2n-1} + y_{2n-2}].$$

Summing both sides of these equations, we find a more general expression for Simpson's Rule.

. . . . . .

$$\int_{x_0}^{x_0 + 2nh} ydx = (h/3)(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{(2n-1)} + y_{2n})$$

The derivation thus far discussed could have been accomplished in a somewhat simpler fashion separately in contrast to their falling out of the same formula, Newton's Binomial Interpolation Formula. However, the power of the use of this one particular formula is in developing additional quadrature formulas by merely increasing the value of n. Each increase of n by unity adds one term to Newton's Formula and assumes the curve to pass through an additional point of data. Furthermore, additional points may be added without destroying previously calculated work as is the case in the method of least squares. Had we included one additional term in our work in calculating Simpson's Rule, thereby assuming the approximation curve to pass through three equidistant points with respect to the independent variable, another approximation formula which is known as the three-eighths rule would have been obtained. Letting $n = 3$ we have,

$$\begin{aligned}\int_{x_0}^{x_0 + 3h} ydx &= h[3y_0 + (9/2)(y_1 - y_0) \\ &\quad + (9 - 9/2)(y_2 - 2y_1 + y_0)/2! \\ &\quad + (81/4 - 27 + 9)(y_3 - 3y_2 + 3y_1 - y_0)/3! \\ &= (3h/8)(y_0 + 3y_1 + 3y_2 + y_3),\end{aligned}$$

or extending by the same principle applied to Simpson's Rule we obtain the more general form

$$\int_{x_0}^{x_0 + 3nh} ydx = (3h/8)[y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots + 3y_{3n-1} + y_{3n}]$$

If we had let $n = 6$ in Newton's Formula after integrating, we would have derived Weddle's Rule for approximate integration which is generally more accurate than Simpson's but requires at least seven consecutive values of the function.

$$\int_{x_0}^{x_0 + 6h} ydx = (3h/10)[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6],$$

or, more generally,

$$\int_{x_0}^{x_0 + 6nh} ydx = (3h/10) \sum_{i=0}^{6n} k_i y_i,$$

where $k_i = 1, 5, 1, 6, 1, 5, 2, 5, 1, 6, 1, 5, 2, \cdots, 5, 1$ as $i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots, 6n - 1, 6n$, respectively.

Other approximate quadrature formulas may be developed for other integral values of n. The value of Newton's Binomial Interpolation Formula to numerical integration is clear. After integrating the formula, it serves as a general expression for the derivation of mechanical quadrature formulas.

"Here and elsewhere we shall not obtain the best insight into things until we actually see them growing from the beginning . . ."

—ARISTOTLE (Politics)

Ruled Surfaces

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It will be recalled that the direction numbers of the line in 3-space from (x_1, y_1, z_1) to (x_2, y_2, z_2) are proportional to $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$; further, that the parametric form of the equations of the line through (x_1, y_1, z_1) with direction numbers (λ, μ, ν) is

$$(1) \qquad \begin{cases} x = x_1 + \lambda p, \\ y = y_1 + \mu p, \\ z = z_1 + \nu p. \end{cases}$$

where p is any real constant. We shall use the equations (1) in developing the algebraic equations of certain ruled surfaces in Cartesian 3-space.

Example 1. To show the techniques used, let us take as the first example the parabolic hyperboloid, or saddle surface. This surface can be generated by joining points of an ordered set of points along a line l to a similarly ordered set of points along a line m skew to line l. Without loss of generality, we shall assume the two directrix lines lie in parallel planes, that these planes are parallel to the YZ-plane, and that the inclinations of the lines to the X-axis are equal in absolute value, but opposite in sign. Then we may write the equations of the directrices as $x = a$; $z = my$ and $x = -a$; $z = -my$. In this case we take as two points on the same ruling and on different directrices $P(a, y, my)$ and $Q(-a, y, -my)$. The direction cosines of PQ are then proportional to a, 0, and my and the parametric form of the equations of a ruling is

$$(2) \qquad \begin{cases} x = a + ap = a(1 + p). \\ y = y. \\ z = my + myp = my(1 + p). \end{cases}$$

From this, we see that

$$x/a = z/my$$

is one form of the equation of the surface or, more simply,

$$(3) \qquad z = cxy.$$

We now show that this surface has (a) two sets of rulings, (b) rulings of the same family are skew, and (c) rulings of different families intersect.

To show (a) we observe that the equations of one set of rulings are

$$\begin{cases} K_1 z = cx, \\ 1 = K_1 y, \end{cases}$$

(4) and that the equations of the other set are

$$\begin{cases} K_2 z = cy, \\ 1 = K_2 x \end{cases}$$

To show (b), we show that the four equations derived from (4) are dependent, *i.e.*, have a common solution.

$$\begin{vmatrix} -c & 0 & K_1 & 0 \\ 0 & -K_1 & 0 & 1 \\ 0 & -c & K_2 & 0 \\ -K_2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -c & 0 & K_1 & 0 \\ K_2 & -K_1 & 0 & 0 \\ 0 & -c & K_2 & 0 \\ -K_2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -c & 0 & K_1 \\ K_2 & -K_1 & 0 \\ 0 & -c & K_2 \end{vmatrix}$$

$$\begin{aligned} &= -c(-K_1)K_2 - K_2(cK_1) \\ &= cK_1K_2 - cK_1K_2 \\ &= 0. \end{aligned}$$

The determinant is zero, independently of K_1 and K_2; so the two rulings of different families must intersect.

When we use two rulings of the same family we get:

$$\begin{cases} Kz = cx, \\ 1 = Ky, \end{cases}$$

(5) and

$$\begin{cases} Hz = cx, \\ 1 = Hy, \end{cases}$$

$$\begin{vmatrix} -c & 0 & K & 0 \\ 0 & -K & 0 & 1 \\ -c & 0 & H & 0 \\ 0 & -H & 0 & 1 \end{vmatrix} = \begin{vmatrix} -c & 0 & K & 0 \\ 0 & -K+H & 0 & 0 \\ -c & 0 & H & 0 \\ 0 & -H & 0 & 1 \end{vmatrix} = \begin{vmatrix} -c & 0 & K \\ 0 & -K+H & 0 \\ -c & 0 & H \end{vmatrix}$$

$$\begin{aligned} &= -c(-KH + H^2) - c(K^2 - KH) \\ &= cKH - cH^2 - cK^2 + cKH \\ &= -c(H^2 - 2KH + K^2) \\ &= -c(H - K)^2. \end{aligned}$$

This is not zero for any distinct choices of K and H, so no two rulings of the same family intersect. (The trivial case $c = 0$, in which the XY-plane is the complete surface, is omitted from the discussion.)

Example 2. As our second example let us take the conoid. The directrix curves are a circle and a line segment whose length is equal to the diameter of the circle. We choose the axes so the equations of the circle are ($x^2 + y^2 = a^2$; $z = m$) and the line is the segment of the X-axis $-a \leqq x \leqq a$.

The parametric equations of the circle are

$$\begin{cases} x = a \cos \theta, \\ y = a \sin \theta, \\ z = m. \end{cases}$$

The equations of the line are

$$\begin{cases} y = 0, \\ z = 0. \end{cases}$$

We now show that if we join a point on the segment with a point on the circle having the same x-coordinate we will produce a conoid whose equation is

$$z^2(a^2 - x^2) = y^2m^2. \tag{6}$$

We take a point on the directrix line segment $P(a \cos \theta, 0, 0)$ and join it to the corresponding point on the directrix circle $Q(a \cos \theta, a \sin \theta, m)$. The direction cosines of PQ are proportional to $[0, a \sin \theta, m]$ and the parametric form of the equations of the ruling can be written as

$$\begin{cases} x = a \cos \theta, \\ y = ap \sin \theta, \\ z = pm. \end{cases}$$

From this

$$\begin{cases} x/a = \cos \theta, \\ y/a = p \sin \theta, \\ z/m = p. \end{cases}$$

Then we get

$$\begin{aligned} x^2/a^2 &= \cos^2\theta, \\ y^2/a^2 &= (z^2/m^2) \sin^2\theta. \\ x^2/a^2 + y^2m^2/a^2z^2 &= 1, \\ z^2x^2 + m^2y^2 &= a^2z^2, \\ z^2(a^2 - x^2) &= m^2y^2. \end{aligned}$$

This establishes (6).

Example 3. As our third example we shall use the elliptic hyperboloid of one sheet. With no loss of generality, assume that the coordinate axes are so chosen that the X-axis and Y-axis are parallel to the axes of an elliptical right section and the centers of all elliptic right sections lie on the Z-axis. Let the centers of two sections be a and b. Then we can say that the directrix ellipses have the equations

$$\begin{cases} x = a \cos \theta, \\ y = b \sin \theta, \\ z = \pm m. \end{cases}$$

These ellipses we will refer to as the bases of the hyperboloid.

If we construct lines joining points on the two bases for which the values of θ are the same, we produce the elliptic cylinder,

$$x^2/a^2 + y^2/b^2 = 1,$$

whose equation is independent of z and whose elements, or rulings, have the equations

$$\begin{cases} x = a \cos \theta, \\ y = b \sin \theta. \end{cases}$$

However, if we join the points $(x \cos \theta, y \sin \theta, -m)$ on the lower base to the points $(x \cos \psi, y \sin \psi, m)$ on the upper base ($\psi - \theta =$ a nonzero constant), we produce an elliptic hyperboloid, whose equation we proceed to develop. We shall show about this figure (a) the equation is $(x^2/\alpha^2) + (y^2/\beta^2) - (z^2/\gamma^2) = 1$; (b) it has two families of rulings; (c) rulings of the same family are skew; (d) rulings of different families intersect.

A point P on one directrix curve may be expressed in parametric form with parameter θ and constant ϕ as $P(a \cos [\theta + \phi], h \sin [\theta + \phi], m)$.
Suppose this point is joined by a ruling to the point

$$Q(a \cos [\theta - \phi], h \sin [\theta - \phi], -m)$$

on the other base. Then the parametric equations of the ruling are

$$\begin{cases} x/a = \cos (\theta + \phi) + p[\cos (\theta + \phi) - \cos (\theta - \phi)], \\ y/b = \sin (\theta + \phi) + p[\sin (\theta + \phi) - \sin (\theta - \phi)], \\ z/m = 1 + 2p \end{cases}$$

From this

$$x^2/a^2 = \cos^2(\theta + \phi) + p^2[\cos^2(\theta + \phi) - 2\cos(\theta + \phi)\cos(\theta - \phi) + \cos^2(\theta - \phi)] + 2p\cos^2(\theta + \phi) - 2p\cos(\theta + \phi)\cos(\theta - \phi).$$

$$y^2/b^2 = \sin^2(\theta + \phi) + p^2[\sin^2(\theta + \phi) - 2\sin(\theta + \phi)\sin(\theta - \phi) + \sin^2(\theta - \phi)] + 2p\sin^2(\theta + \phi) - 2p\sin(\theta + \phi)\sin(\theta - \phi).$$

$$x^2/a^2 + y^2/b^2 = 1 + p^2[1 + 1 - 2\cos 2\phi] + 2p[1 - \cos 2\phi]$$
$$= 1 + 2p(1 - \cos 2\phi) + 2p^2(1 - \cos 2\phi).$$

$$z^2/m^2 = 1 + 4p + 4p^2.$$

$$2x^2/a^2 + 2y^2/b^2 = 1 + [1 + 4p(1 - \cos 2\phi) + 4p^2(1 - \cos 2\phi)]$$
$$= 1 + (1 - \cos 2\phi)[1 + 4p + 4p^2] + (\cos 2\phi).$$

$$2x^2/[a^2(1 + \cos 2\phi)] + 2y^2/[b^2(1 \cos 2\phi)] - z^2(1 - \cos 2\phi)/[m^2(1 + \cos 2\phi)] = 1, \text{ or}$$

$$x^2/\alpha^2 + y^2/\beta^2 - z^2/\gamma^2 = 1 \tag{7}$$

This is one form of the equation of the surface. The interpretation of ϕ is that it is half the difference in phase of the parameter in the representation of the two elliptic bases when $z = m$ and $z = -m$. If $\phi = 0$, we have $x^2/a^2 + y^2/b^2 = 1$, which is the elliptic cylinder.

It will be seen that the numbers α, β, and γ depend on the choice of ϕ for any fixed a, b, and m, but the form of the equation is not affected by the choice of ϕ. The geometric interpretations of the numbers α, β, and γ are these:

α and β are the semimajor and semiminor axes of the elliptic section of the surface made by the XY-plane, which in this case is the minimum section.

γ is the semiconjugate axis and α (or β) the semitransverse axis of the hyperbolic section of the surface by the XZ-plane (or YZ-plane).

The surface is symmetric with respect to the origin and to each of the coordinate axes and planes. The x-intercepts and y-intercepts

are $\pm a$, and $\pm b$ respectively. There is no z-intercept. The ellipse and the two hyperbolas

$$\begin{aligned} x^2/\alpha^2 + y^2/\beta^2 &= 1,\ z = 0, \\ x^2/\alpha^2 - z^2/\gamma^2 &= 1,\ y = 0, \\ y^2/\beta^2 - z^2/\gamma^2 &= 1,\ x = 0, \end{aligned}$$

are the traces in the XY-plane, XZ-plane, and YZ-plane.

The section in the plane $y = \beta$ is the pair of lines $\gamma x \pm \alpha z = 0$, $y = \beta$. Any other plane $y = k$ cuts from the surface a hyperbola whose orientation depends on whether $k^2 < \beta^2$ or $k^2 > \beta^2$.

Sections perpendicular to the Z-axis are ellipses.

The surface is composed of only one sheet; it is possible to pass from any point on the surface to any other point on the surface without leaving the surface.

If $\alpha = \beta$, the elliptic sections become circles, and the surface is a hyperboloid of revolution of one sheet.

The equations of the families of rulings of the elliptic hyperboloid are written in the form

$$(8) \qquad \begin{cases} K_1(x/\alpha - z/\gamma) = 1 + y/\beta \\ x/\alpha + z/\gamma = K_1(1 - y/\beta) \end{cases} \text{and}$$

$$\begin{cases} K_2(x/\alpha - z/\gamma) = 1 - y/\beta \\ x/\alpha + z/\gamma = K_2(1 + y/\beta) \end{cases}$$

From these equations we prove the properties of rulings mentioned earlier, employing techniques used twice before in this paper. The determinant

$$\begin{vmatrix} K_1/\alpha & -1/\beta & -K_1/\gamma & -1 \\ 1/\alpha & K_1/\beta & 1/\gamma & -K_1 \\ K_2/\alpha & 1/\beta & -K_2/\gamma & -1 \\ 1/\alpha & -K_2/\beta & 1/\gamma & -K_2 \end{vmatrix}$$

$$= K_1/\alpha\beta\gamma \begin{vmatrix} K_1 & 1 & -K_1 \\ 1 & -K_2 & -1 \\ -K_2 & 1 & -K_2 \end{vmatrix} - 1/\alpha\beta\gamma \begin{vmatrix} -1 & -K_1 & -1 \\ 1 & -K_2 & -1 \\ -K_2 & 1 & -K_2 \end{vmatrix}$$

$$+ K_2/\alpha\beta\gamma \begin{vmatrix} -1 & -K_1 & -1 \\ K_1 & 1 & -K_1 \\ -K_2 & 1 & -K_2 \end{vmatrix} - 1/\alpha\beta\gamma \begin{vmatrix} -1 & -K_1 & -1 \\ K_1 & 1 & -K_1 \\ 1 & -K_2 & -1 \end{vmatrix}$$

$$= (K_1/\alpha\beta\gamma)(2K_1K_2^2 + 2K_2) + (1/\alpha\beta\gamma)(2 + 2K_1K_2)$$
$$+ (K_2/\alpha\beta\gamma)(-2K_1 - 2K_1^2K_2) - (1/\alpha\beta\gamma)(2 + 2K_1K_2) = 0$$

The determinant equals zero independently of K_1 and K_2. This shows that two rulings of different families must intersect.

Consider the determinant of the matrix of the system derived from equations of two pairs of rulings of the same family, say

$$\begin{cases} K(x/\alpha - z/\gamma) = 1 + y/\beta \\ x/\alpha + z/\gamma = K(1 - y/\beta) \end{cases} \text{and} \begin{cases} H(x/\alpha - z/\gamma) = 1 + y/\beta \\ x/\alpha + z/\gamma = H(1 - y/\beta) \end{cases}$$

$$\begin{vmatrix} K/\alpha & -1/\beta & -K/\gamma & -1 \\ 1/\alpha & K/\beta & 1/\gamma & -K \\ H/\alpha & -1/\beta & -H/\gamma & -1 \\ 1/\alpha & H/\beta & 1/\gamma & -H \end{vmatrix}$$

$$= K/\alpha\beta\gamma \begin{vmatrix} K & 1 & -K \\ -1 & -H & -1 \\ H & 1 & -H \end{vmatrix} - 1/\alpha\gamma\beta \begin{vmatrix} -1 & -K & -1 \\ -1 & -H & -1 \\ H & 1 & -H \end{vmatrix}$$

$$+ H/\alpha\beta\gamma \begin{vmatrix} -1 & -K & -1 \\ K & 1 & -K \\ H & 1 & -H \end{vmatrix} - 1/\alpha\gamma\beta \begin{vmatrix} -1 & -K & -1 \\ K & 1 & -K \\ -1 & -H & -1 \end{vmatrix}$$

$$\begin{aligned} &= (K/\alpha\beta\gamma)(2K - 2H) - (1/\alpha\beta\gamma)(2KH - 2H^2) \\ &+ (H/\alpha\beta\gamma)(2H - 2K) - (1/\alpha\beta\gamma)(-2K^2 + 2KH) \\ &= (4/\alpha\beta\gamma)(K - H)^2. \end{aligned}$$

This is not zero for any distinct choices of K and H. Since this is so, no two rulings of the same family intersect.

"And Lucy, dear child, mind your arithmetic • • •. What would life be without arithmetic, but a scene of horrors?"

—SIDNEY SMITH

Relations Between Hyperbolic and Circular Functions

RICHARD FRANKE
Student, Fort Hays Kansas State College

The hyperbolic functions are familiar to all those who have taken very much mathematics. There are several interesting relationships between hyperbolic functions and circular functions which are not generally taught due to time limitations. We know that the hyperbolic functions bear a relationship to the equilateral hyperbola which is very similar to the relationship the circular functions bear to the circle.

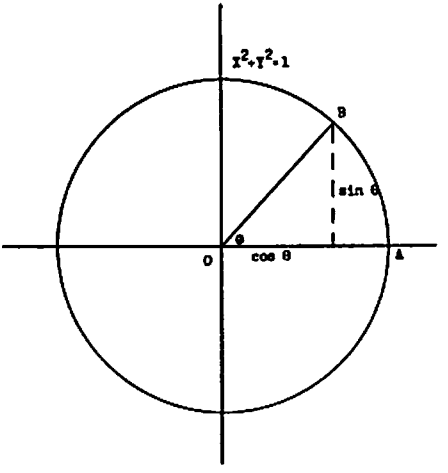

Figure 1

We have the circular functions, sin Θ and cos Θ, related to a unit circle as shown in Figure 1. The area of the sector *OAB* is $K_c = \Theta/2$ or $\Theta = 2K_c$.

It will now be shown that the parameter, u, in the hyperbolic functions is also twice the area *OAB* as shown in Figure 2. The area

of ODB less the area of ADB will be equal to the area of OAB.

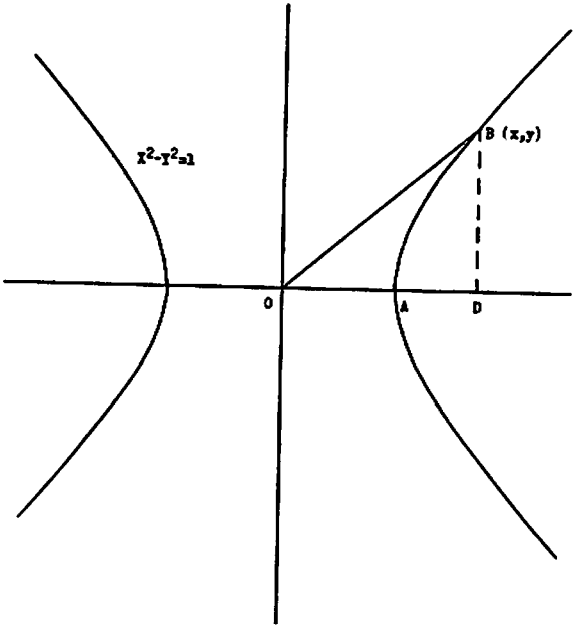

Figure 2

$$
\begin{aligned}
K_h &= xy/2 - \int_1^x \sqrt{x^2 - 1}\,dx \\
&= xy/2 - \tfrac{1}{2}[x\sqrt{x^2 - 1} - ln(x + \sqrt{x^2 - 1})]_1^x \\
&= xy/2 - \tfrac{1}{2}[x\sqrt{x^2 - 1} - ln(x + \sqrt{x^2 - 1})] \\
&= \tfrac{1}{2}\,ln(x + \sqrt{x^2 - 1})
\end{aligned}
$$

If $x = \cosh u$ and $y = \sinh u$, then $x^2 - y^2 = \cosh^2 u - \sinh^2 u$. By definition, $\cosh u = (e^u + e^{-u})/2$ and $\sinh u = (e^u - e^{-u})/2$. Then $x^2 - y^2 = 1$, which is an equilateral hyperbola (Figure 2).

Since $x = \cosh u = (e^u + e^{-u})/2$, it follows that $e^{2u} - 2xe^u + 1 = 0$. Solving this equation as a quadratic in e^u,

$$e^u = (2x + \sqrt{4x^2 - 4})/2 = x + \sqrt{x^2 - 1}$$

Solving for u, $\qquad u = \ln(x + \sqrt{x^2 - 1})$.

Comparing this with the area, K_h, of the sector, we see that $u = 2K_h$.

We can now note that the two parameters, u and Θ, are respectively equal to twice the areas of their sectors.

By drawing the hyperbolic and circular functions on the same graph, we may note several interesting things (Figure 3).

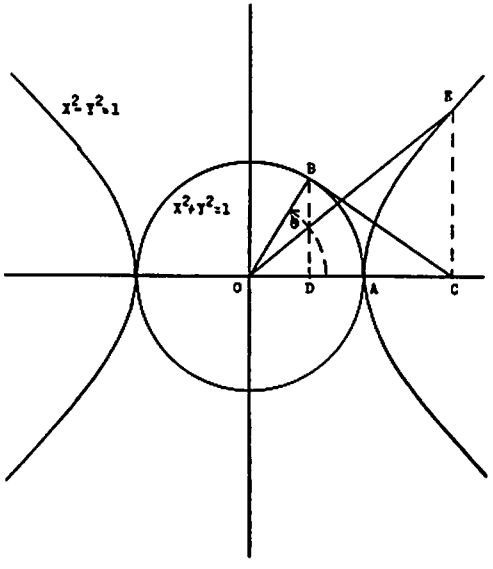

Figure 3

Area $OAE = \frac{1}{2}u$.
$OD = \cos \Theta$.

$OD/OB = OB/OC$, or $OD \cdot OC = 1$, since $OB = 1$.
$\cos \Theta \cdot OC = 1$.
$OC = 1/\cos \Theta = \sec \Theta$.
$OC = \cosh u = \sec \Theta$.

Then $\sqrt{\sinh^2 u + 1} = \sqrt{\tan^2 \Theta + 1}$, and $\sinh u = \tan \Theta$.

By similar substitutions, we arrive at the following relationships:

$$\begin{aligned} \tanh u &= \sin \Theta, \\ \coth u &= \csc \Theta, \\ \operatorname{csch} u &= \cot \Theta, \\ \operatorname{sech} u &= \cos \Theta. \end{aligned}$$

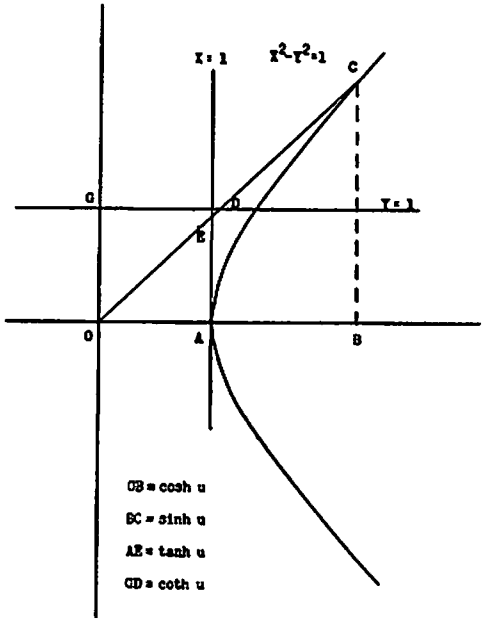

Figure 4

In these relations the variable Θ is called the gudermannian of u, and is symbolized $\Theta = gd\ u$. The value of u may be found in terms of functions of Θ quite easily.

$$\sinh u = \tan \Theta.$$

Taking the inverse hyperbolic sine of both sides,

$$u = \sinh^{-1}(\tan\theta) = \ln(\tan\theta + \sqrt{\tan^2\theta + 1})$$
$$= \ln(\tan\theta + \sec\theta).$$

$$u = \ln(\tan gd\, u + \sec gd\, u)$$

It is interesting to note that some of the hyperbolic functions can be read directly from a graph in a manner similar to that of reading the circular functions. (Figure 4).

$$BC/OB = AE/OA$$
$$\sinh u/\cosh u = AE = \tanh u$$
$$OB/BC = GD/OG$$
$$\cosh u/\sinh u = GD = \coth u$$

The series representations of the hyperbolic and circular functions also show a close relationship. The series for the hyperbolic sine is found very easily through the exponential definition of it and the well-known series for e^x. The series is:

$$\sinh x = x + x^3/3! + x^5/5! + x^7/7! + \cdots.$$

The sin x series should be familiar to all of us and can be found rather easily with Maclaurin's expansions. It is:

$$\sin x = x - x^3/3! + x^5/5! - x^7/7! + \cdots.$$

It can be seen that the two series differ only in the signs. It is also seen that

$$\sin ix = ix - (ix)^3/3! + (ix)^5/5! - (ix)^7/7! + \cdots$$
$$= i(x + x^3/3! + x^5/5! + x^7/7! + \cdots),$$
$$\text{or } \sin ix = i \sinh x.$$

By treating the series for other circular and hyperbolic functions similarly, we find these relationships:

$$\sinh(ix) = i \sin x$$
$$\cos(ix) = \cosh x$$
$$\cosh(ix) = \cos x$$
$$\tan(ix) = i \tanh x$$
$$\tanh(ix) = i \tan x$$

It is interesting to note that the circular functions can be reduced to exponential functions.

$$\sinh(ix) = (e^{ix} - e^{-ix})/2 = i \sin x;$$
$$\sin x = (e^{ix} - e^{-ix})/2i.$$
$$\cosh(ix) = (e^{ix} + e^{-ix})/2 = \cos x;$$
$$\cos x = (e^{ix} + e^{-ix})/2.$$

References:

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Neeley, John H., and J. I. Tracey, *Differential and Integral Calculus*, Macmillan Book Company, 1954.

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Standard Mathematical Tables, Chemical Rubber Publishing Company, Cleveland, Ohio, 1957.

"Someone who had begun to read geometry with Euclid, when he had learned the first proposition, asked Euclid, 'But what shall I get by learning these things?' whereupon Euclid called his slave and said, 'Give him three-pence since he must make gain out of what he learns.' "

—STOBAEUS

The Problem Corner

EDITED BY J. D. HAGGARD

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before October 1, 1959. The best solutions submitted by students will be published in the Fall, 1959, issue of THE PENTAGON, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to J. D. Haggard, Department of Mathematics, Kansas State College, Pittsburg, Kansas.

PROPOSED PROBLEMS

121. *Proposed by the Editor* (From a Russian university entrance examination).

If A, B and C are the angles formed by a diagonal of a rectangular parallelepiped with its edges, show that:

$$\sin^2 A + \sin^2 B + \sin^2 C = 2.$$

122. *Proposed by George Mycroft, Kansas State College, Pittsburg.*

With each letter symbolizing a digit decode the following puzzle:

$$\begin{array}{r} \text{FIVE} \\ -\ \text{FOUR} \\ \hline \text{ONE} \\ +\ \text{ONE} \\ \hline \text{TWO} \end{array}$$

123. *Proposed by the Editor* (From *The American Mathematical Monthly*).

Let a real positive number n be split into x equal parts in such a manner that the product of the parts will be greatest. How many parts will there be?

124. *Proposed by the Editor.*

Show that the square of any odd integer is one more than an integral multiple of eight.

125. *Proposed by the Editor* (From a Russian university entrance examination).

Two factories each received an order for an identical number of machines. The first factory started 20 days earlier and finished work 5 days earlier than the second factory. At the moment when the number of machines made by both factories taken together was equal to one-third of the total number on order, the number of machines made by the first factory was four times the number produced by the second.

The first factory worked on the order altogether x days, producing m machines per day; the second factory worked y days, producing n machines per day. Find those of the quantities x, y, m, and n and those of the ratios x/y and m/n which can be determined from the data given in the problem.

SOLUTIONS

116. *Proposed by J. Max Stein, student, Colorado State University.*

While traveling in Iowa in the spring, I observed that there were several directions I could look across a field of check-planted corn with the hills of corn apparently lying along straight lines. Assuming that a hill of corn determines a point, that the distance between rows is the same as the distance between hills in a row, and that the field is infinite in extent then by looking over any hill of corn will there be any direction I can look across the field such that my line of sight will contain no other hill of corn?

Solution by Paul R. Chernoff, Central High School, Philadelphia, Pennsylvania.

Take the distance between adjacent hills of corn as unity. Then the field can be thought of as the Cartesian plane, with the hills of corn represented by the points of a lattice, the coordinates of which are integers referred to any one point of the lattice as the origin. The problem then becomes one of passing a line through the origin such that it will pass through no other lattice point. This can be done by taking the inclination, Θ, of the line such that $\tan \Theta$ is irrational. For if this line did pass through another point of the lattice, then the ratio of the ordinate to the abscissa of that point would equal $\tan \Theta$. But this implies that $\tan \Theta$ can be expressed as a rational number, which contradicts our initial condition that $\tan \Theta$ is irrational.

Also solved by Mark Bridger, High School of Science, Bronx, New York; Warren Shreve, Iowa State Teachers College, Cedar Falls, Iowa.

117. *Proposed by Frank Hawthorne, New York State Department of Education.*

(From a New York Regents examination in high school algebra).

A man traveled 60 miles by bus and 600 miles by plane, taking 6 hours for the trip. On the return trip the speed of the plane was reduced by 50 miles an hour, but the speed of the bus was increased by 10 miles per hour so that the return trip also took 6 hours. Find the average speed of the plane and the average speed of the bus.

Solution by Dick Barnett, Nebraska State Teachers College, Wayne, Nebraska.

Let x = speed of bus in mph going,

$x + 10$ = speed of bus in mph returning,

y = speed of plane in mph going,

$y - 50$ = speed of plane in mph returning.

(1) $(60/x) + (600/y) = 6$, for trip going,

(2) $60/(x + 10) + 600/(y - 50) = 6$, for return trip.

Simplifying (1) and (2) we obtain:

(3) $10y + 100x - xy = 0$,

(4) $150x - xy = -1000$.

Eliminating x from (3) and (4) we obtain $y^2 - 250y + 10{,}000 = 0$ whose roots are $y = 50$ and 200. Substituting into (3) gives $x = -10$ and 20. $y = 50$ and $x = -10$ must be discarded.

Thus, speed of bus going = 20 mph,

speed of plane going = 200 mph,

speed of bus returning = 30 mph,

speed of plane returning = 150 mph.

Substituting these results into the formula $t = d/r$ gives

3 hours for bus going,

3 hours for plane going,

2 hours for bus returning,

4 hours for plane returning.

The total time of the bus is 5 hours and the total distance is 120 miles. Thus the average speed is 24 mph.

The total time of the plane is 7 hours and the total distance is 1200 miles. Thus the average speed is 171.43 mph.

Also solved by Bob Stafford, Wake Forest College, Winston-Salem, North Carolina; Paul R. Chernoff, Central High School, Philadelphia, Pennsylvania; Fred Barber, Wayne State University, Detroit, Michigan; Mark Bridger, High School of Science, Bronx, New York; and Mary Sworske, Mount Mary College, Milwaukee, Wisconsin.

118. *Proposed by Frank C. Gentry, University of New Mexico, Albuquerque.*

Find four integers a, b, c, d; $a < b < 10$, $c < 10$, $d < 10$, such that $(a/b)(10c + d) = 10d + c$.

Solution by Philip Farber, Wayne State University, Detroit, Michigan.

$$(a/b) = (10d + c)/(10c + d) = (10d/c + 1)/(10 + d/c)$$

Since the last fraction must reduce to one with numerator and denominator less than 10, namely a/b, $10d/c + 1$ and $10 + d/c$ are not relatively prime. Therefore $d/c \neq 1, 3, 4, 5, 6, 7, 9$; thus $d/c = 2$ or 8. From the symmetrical role of d and c it is obvious $c/d = 2$ or 8. But $d/c = 2$ or 8 would make $b < a$. Therefore $c/d = 2$ or 8. We may therefore have (1) $c = 2$, $d = 1$; or (2) $c = 4$, $d = 2$; or (3) $c = 6$, $d = 3$; or (4) $c = 8$, $d = 4$. For each of these cases $c/d = 2$ thus $a/b = 4/7$, giving $a = 4$ and $b = 7$. For $c/d = 8$, we have (5) $c = 8$, $d = 1$, giving $a/b = 2/9$, thus $a = 2$, $b = 9$.

A tabulation of the solutions is as follows:

a	=	4	4	4	4	2
b	=	7	7	7	7	9
c	=	2	4	6	8	8
d	=	1	2	3	4	1

Also solved by Fred Barber, Wayne State University, Detroit, Michigan; Mark Bridger, High School of Science, Bronx, New York.

119. *Proposed by Mark Bridger, High School of Science, Bronx, New York.*

Find the sum of all numbers greater than 10,000 formed by using the digits 0, 2, 4, 6, 8, no digit being repeated in any number.

Solution by Fred Barber, Wayne State University, Detroit, Michigan.

There are $P(4, 4) = 4! = 24$ different arrangements of the nonzero units digits. Six of these have the leading digit zero and thus are to be discarded. Hence the sum of all the digits in the units place will be $18(0 + 2 + 4 + 6 + 8) = 18(20)$. The same is true for the sum of the digits in the tens, hundreds, and thousands places; but since zero cannot be used in the leading position there are $P(4, 4) = 24$ choices for this position. Therefore the sum of all the numbers is

$$= 18(20)(1) + 18(20)(10) + 18(20)(100) + 18(20)(1000) + 24(20)(10{,}000)$$

$$= 20\left(\sum_{n=0}^{3} 18\cdot 10^n + 24\cdot 10^4\right) = 5{,}199{,}960$$

Also solved by Philip Farber, Wayne State University, Detroit, Michigan.

120. *Proposed by Problem Corner Editor.* (Taken from the July, 1957, Preliminary Actuarial Examination).

If $g'(x) = f(x)$ and $f(x)$ is continuous, evaluate

$$\int_a^b f(x)g(x)dx.$$

1. *Solution by Mark Bridger, High School of Science, Bronx, New York.*

Let $y = g(x)$, then $dy = f(x)\,dx$, and

$$\int_a^b f(x)g(x)dx = \int_{g(a)}^{g(b)} y\,dy = [y^2/2]_{g(a)}^{g(b)} = \tfrac{1}{2}[g^2(b) - g^2(a)]$$

Since $f(x)$ is continuous on the interval (a,b), and is equal to $g'(x)$, thus $g'(x)$ is continuous. The continuity of $g'(x)$ implies $g(x)$ is continuous; thus $g(a)$ and $g(b)$ are both defined, and the integral may be evaluated as indicated.

2. *Solution by Bob Stafford, Wake Forest College, Winston-Salem, North Carolina.*

Since $g'(x) = f(x)$, $\displaystyle\int_a^b f(x)g(x)dx = \int_a^b g(x)g'(x)dx.$

Using the formula for integration by parts ($\int u\,dv = uv - \int v\,du$) and letting $u = g(x)$ and $dv = g'(x)dx$, we obtain $du = g'(x)dx$ and $v = g(x)$.

$$\int_a^b g(x)\,g'(x)dx = [g^2(x)]_a^b - \int_a^b g(x)g'(x)dx$$

$$\text{or} \int_a^b f(x)g(x)dx = \tfrac{1}{2}[g^2(x)]_a^b = \tfrac{1}{2}[g^2(b) - g^2(a)].$$

Also solved by Warren E. Shreve, Iowa State Teachers College, Cedar Falls, Iowa; Brenda Fried, Hofstra College, Hempstead, New York; Paul R. Chernoff, Central High School, Philadelphia, Pennsylvania; Dick Barnett, Nebraska State Teachers College, Wayne, Nebraska.

"Pure mathematics consists entirely of such associations as that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true • • •. If our hypothesis is about *anything* and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

—Bertrand Russell

THE MATHEMATICAL SCRAPBOOK

EDITED BY J. M. SACHS

There are only two ways open to man for attaining a certain knowledge of truth: clear intuition and necessary deduction.

—DESCARTES

=△=

Postulate deduction, an important aspect of postulational thinking, is not a panacea—nothing is that—but it is essential as a means to many good ends: it increases knowledge in all fields; it is a powerful instrument for doctrinal criticism; it shows us how hard it is to know; it fosters scientific modesty, discourages dogmatism, favors tolerance, and makes for the maintenance and advancement of good will in the world.

—C. J. KEYSER

=△=

The preceding quotation came to the attention of the Editor quite a while ago. It was recalled recently upon reading a column in a newspaper meant to be both humorous and critical. The columnist wrote, "I read recently of a study by a psychologist which seemed to indicate that good spellers are reticent. Just this week I received a folder of letters written, as an assignment, by a class in one of our local schools. An examination of the letters convinces me that a reticent child would be pretty lonely in the school."

Can you ferret out the postulates? What do you think of the logic behind this wisecrack? I tried the same "postulate deduction" on the editorials in the same paper. This was a more difficult task but quite rewarding. Try it and compare notes with someone else who has done the same thing independently. The results may amaze you, and they are almost certain to convince you that understanding one another is not a simple matter.

=△=

Mr. A was inordinately proud of his enormous fast car. His friend Mr. B was equally proud of his tiny economical car. Leaving from the same place at the same time, they drove to Lake Blank for a holiday. Discussing the trip at breakfast the following day, each

boasted of the merits of his particular car as shown by the trip. From their claims can you discover the distance to Lake Blank from the starting point, the average speed maintained by each, the time for the trip for each, the number of gallons of gasoline used by each?

Mr. A: If I had driven 5 miles per hour faster, and I could have with my car, I would have been here 3 hours before you.

Mr. B: If I had used twice as much gas as I did, I would still have used 20 gallons less than you did.

Mr. A: If only I had driven the 5 miles per hour faster my average rate would have been 60% greater than yours.

Mr. B: If we had been going in opposite directions, we would have been separated by the distance to Lake Blank in 3 hours and 12 minutes.

Mr. A: If I had driven 5 miles per hour faster, the sum of the number of hours for my trip and the number of gallons of gas I actually used would have been 27 greater than the sum of the number of hours for your trip and the number of gallons you used.

=△=

Now the first noticeable fact about arithmetic is that it applies to everything, to tastes and to sounds, to apples and to angels, to the ideas of the mind and to the bones of the body. The nature of the things is perfectly indifferent, of all things it is true that two and two make four. Thus we write down as the leading characteristic of mathematics that it deals with properties and ideas which are applicable to things just because they are things, and apart from any particular feelings, or emotions, or sensations, in any way connected with them. This is what is meant by calling mathematics an abstract science.

—A. N. WHITEHEAD

=△=

Even though the invention of the idea of decimals probably preceded the publication of the *La Disme* of Simon Stevin in 1585, his work, clearly illustrating this important advance did not create a revolution in the arithmetic of his day. He used small circles to indicate decimal places. Variations on this rather clumsy device included notations using parentheses, Roman numerals to identify all places

or just the last place, a gap for a decimal point, a series of points, and many others: 27.15; 27⓪1①5②; 271(1)5(2); 27 $\overset{\shortparallel}{15}$; 27.$\overset{\shortparallel}{1}$. $\overset{\shortparallel}{5}$; 27|$\underline{15}$; 27:15; 27,15".

A wide variety of notations were adopted and almost as many were discarded rather quickly. The most persistent separators of the integral and fractional parts were the comma and the period. In the 1700's the comma gained favor on the continent of Europe. This may have been so because the period had already been adopted, at the suggestion of Leibniz, as a symbol for multiplication. In England the period was used for both, ambiguity being avoided by an agreement that 27.15 is 405 while 27·15 is 27 and 15/100.

Isaac Greenwood, in the first arithmetic printed in the United States, suggested either a comma or a period for the decimal point but used the comma himself. Other later books in this country used the raised period as in England. This innovation was soon discarded however, and for a while the period was used for both the multiplication and decimal point signs in the same position. Eventually the distinction was made by an agreement exactly opposite to the one made in England.

On the continent of Europe the comma is still used in some places and the raised period in others as a decimal point. Where the comma is used as a decimal separator, the period is often used in large numbers as we use the comma, *i.e.*, 1,000,000 may be written as 1.000.000.

=△=

The physicist needs his mathematics; the chemist rests more and more on physics; and so it is no wonder that the biologist and medical man have turned to physics and chemistry for further inspiration.

—B. Harrow

=△=

In modern times the belief that the ultimate explanation of all things was to be found in Newtonian mechanics was an adumbration of the truth that all science as it grows towards perfection becomes mathematical in its ideas.

—A. N. Whitehead

The theory of probability seems to have its modern roots in some 17th century correspondence between Fermat and Pascal about the chances of the players in a gambling game. In the same century, Huygens, who is perhaps better known for his contributions to physics, wrote a systematic work on probability. Jacob Bernoulli, at the start of the 16th century, included Huygens' work plus his own comments thereon and a treatment of permutations and combinations in his *Ars Conjectandi.*

Daniel Bernoulli, Jacob's nephew, wrote widely on probability and proposed a theory of "moral expectation" to supercede "mathematical expectation." Abraham De Moivre, French-born mathematician, wrote, in his adopted homeland, England, such works on probability as *The Doctrine of Chance,* published in 1716. In the 18th century, D'Alembert turned a critical eye upon the foundations of the theory of probability. Many of his countrymen concerned themselves with widely diverse aspects of the subject. De Condorcet did a number of investigations in connection with voting and came to a conclusion which may seem obvious today, that is, the more enlightened the voters the greater the probability of a *correct* judgment. (A considerable amount of political probability was done quite early in the history of the subject.) De Condorcet further proposed the elimination of capital punishment. He argued that although the probability of each judgment being correct might be high that the probability of an incorrect judgment among many judgments was also high, *i.e.,* it was highly probable that at least one person would be executed when he was not guilty.

At the beginning of the 19th century Lagrange and Laplace published important works on probability, discussing both the philosophical basis as well as the mathematical techniques. The treatment of least squares by Laplace was hotly criticized by James Ivory, a 19th century Scot who suggested alternative approaches not dependent upon probability. Unfortunately his alternatives left much to be desired.

In the 19th century the Academy of Paris devoted some time to a discussion of the problem of settling questions of morality by the use of probability. There was much spirited argument as to whether the infant science was applicable in such an area. In this argument a number of eminent scientists and philosophers of several nations participated.

"The probability of getting heads in one toss of an honest coin," says A, "is ½. If I toss two such coins simultaneously the probability of getting a head on the first is ½ and the probability of getting a head on the second is ½ so that the probability of getting a head when the two coins are tossed is ½ + ½ or 1. A similar argument may be made for a tail. Thus whenever two coins are tossed it is certain that there will be one head and one tail."

"Sounds fishy to me," says B, tossing two coins which happen to land with heads showing on both.

What do you think about it?

=△=

A plane leaves city X for city Y at the same time that another plane is leaving city Y bound for X. (For the moment let others worry about the meaning of simultaneity.) They follow roughly the same path, the plane bound for Y reaching that city 1 hour and 20 minutes after they pass one another and the plane bound for city X reaching there 3 hours after they pass one another. (For the moment let others worry about plane crashes.) How do their speeds compare?

=△=

One of the chief triumphs of modern mathematics consists in having discovered what mathematics really is.

—B. Russell

=△=

"I have a very interesting 1959 automobile license," Professor Devious told his class. "It consists of the letter H followed by a dash followed by four digits, none of them zero. If I turn it upside down it is still a number, now followed by the letter H. The original number is 1278 larger than the number you get when the plate is turned upside down. In the original number there are no two digits alike and the units digit is smaller than the thousands digit. Can you," he asked his class, "tell me my license number without sneaking out to look at my car?"

Admission to its sanctuary (astronomy) and to the privileges and feelings of a votary, is only to be gained by one means—sound and sufficient knowledge of mathematics, the great instrument of all exact inquiry, without which no man can ever make such advances in this or any other of the higher departments of science as can entitle him to form an independent opinion on any subject of discussion within their range.

—J. HERSCHEL

The Book Shelf

EDITED BY R. H. MOORMAN

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of THE PENTAGON. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor R. H. Moorman, Box 169A, Tennessee Polytechnic Institute, Cookeville, Tennessee.

Mathematics of Physics and Modern Engineering, I. S. Sokolnikoff and R. M. Redheffer, McGraw-Hill Book Company, Inc. (330 West 42nd Street) New York, 1958, 775 pp., $9.50.

This book is a substantial expansion of *Higher Mathematics for Engineers and Physicists* which was also co-authored by Professor Sokolnikoff. The style is the same good clear exposition as that in the earlier book. It includes just about every elementary applied problem in the physical sciences—and might almost be considered an encyclopedia in this field.

The authors state that their objective is to present a self-contained (only elementary calculus is assumed) introduction to applied mathematics which is both "sound and inspiring." One uninspiring aspect of the book is the number of pages—775. However, the reviewer feels that a desirable balance between intuitive and mathematical approach is reached.

The book contains nine long chapters—each of which is complete and virtually independent of the others. The titles are: Ordinary Differential Equations, Infinite Series, Functions of Several Variables, Algebra and Geometry of Vectors and Matrices, Vector Field Theory, Partial Differential Equations, Complex Variables, Probability, and Numerical Analysis. It also includes appendices: Determinants, Laplace Transforms, and Comparison of Riemann and Lebesgue Integrals.

There are many potential uses for this book. An introductory one-year course in applied mathematics could be developed around material from the earlier parts of the various chapters. The book has sufficient material for at least four consecutive one-semester courses meeting three hours a week. It could be used for one-semester courses

in Infinite Series, Ordinary Differential Equations, Partial Differential Equations, Vector Analysis, Applied Advanced Calculus, Complex Variables, and Probability and Numerical Methods. It could also serve as a good elementary reference in any of these subjects. The material is very readable and the problems are well chosen. There are many good illustrative examples and diagrams throughout.

Considering the vastness of the book it is difficult to appraise a representative set of topics. Two topics did catch the reviewer's eye. One is the meaningful introductory presentation of the Dirac-delta function. The other is the intuitive approach to Lebesgue integrals which, although sketchy, presents the basic idea of measure and Lebesgue integration in a manner which should be easy for the elementary student to understand.

The authors state, "The contents of this book include what we believe should be the minimum mathematical equipment of a scientific engineer." Because of this, as well as the excellent method of presentation, the reviewer highly recommends this book to any one interested in applied mathematics.

—G. Olive
Anderson College

Mathematics for Science and Engineering, Philip L. Alger, McGraw-Hill Book Company, Inc. (330 West 42nd Street) New York, N. Y., 1957, 360 pp., $5.50.

This book is a revision of *Engineering Mathematics* by Charles P. Steinmetz which was published in three editions between 1911 and 1917. There are chapters on arithmetic, trigonometry, directed numbers, algebraic equations, infinite series, numerical calculations, empirical curves, differentials and integrals, functions, trigonometric series, maxima and minima, differential equations, probability, and mathematical models and electric circuits. From the chapter headings one sees all of the traditional undergraduate mathematics in an engineering program. The material is well written and a person with only a limited background could probably go through it.

The material is quite traditional and would not in any way satisfy the people who are interested in seeing more of the modern concepts in mathematics introduced into the curriculum. The nearest thing to the modern concepts would be several paragraphs near the end, one of which is on Boolean algebra.

The number of problems given for the students to solve is quite limited. The maximum number for any one chapter is thirty-six and the minimum number is zero. Those that are given are well chosen.

The figures in this book are numerous and very good which makes for better understanding of the text material. The bibliography given at the end of each chapter is sufficient for further reading. The appendixes run from A to I and include the usual tables.

This book could best be used by the student that is interested in reviewing the traditional undergraduate mathematics program from an engineering point of view.

—IVEY C. GENTRY
Wake Forest College

Vector Analysis, Louis Brand, John Wiley and Sons, Inc. (440 Fourth Avenue) New York 16, N. Y., 1957, 270 pp., $6.00.

Vector Analysis by Louis Brand presents clearly and concisely the fundamental theory of vector algebra and vector calculus as well as numerous fully illustrated applications in many fields, including modern geometry and mathematical physics. Because of its simplicity and relative completeness, the book will no doubt prove useful as an introductory textbook, a handbook, or a library reference.

The well-known teacher and author, Professor Brand, gives encouragement to the beginning student by vividly illustrating the advantages of using vectors in the solution of problems such as finding the distance of a point from a plane. This problem is readily solved without using the equation of the plane.

The first five chapters cover the fundamentals: vector algebra, line vectors, vector functions of one variable, differential invariants (with the introduction here of divergence and rotation as invariants of a dyadic), and integral theorems. Included is a thorough coverage of such topics as line integrals, Green's theorem in the plane, generalized Stokes' theorem, and Green's identities. Continuity of fundamental theory is achieved by leaving as exercises the proofs of such theorems as Desargues' theorem and Kelvin's minimum-energy theorem.

Following these five introductory chapters are three chapters which are devoted to dynamics, fluid mechanics, and a logical development of electrodynamics based on Maxwell's equations. A final chapter gives a brief introduction to vector spaces (including Hilbert space), with short sections given to the discussion of linear dependence, basis, and dimension.

Vector Analysis by Professor Brand will prove stimulating for anyone who has an interest in the study of mathematics. The reviewer believes, however, that careful study by the beginning student of this brief but comprehensive presentation should contribute greatly to his mastery of the tools of vector algebra and vector calculus and should give him a broad insight into their manifold applications.

—EMMA BREEDLOVE SMITH
Virginia State College

An Introduction to the Foundations and Fundamental Concepts of Mathematics, Harold Eves and Carroll V. Newsom, Rinehart and Company, Inc. (232 Madison Avenue), New York, 1958, xv + 363 pp., $6.75.

This is an important book, an addition to mathematical literature that has long been needed. It is the most readable of the numerous publications on the foundations of mathematics that have appeared in the past few years. A mathematics major who is a junior, a senior, or even a graduate student will find his time well spent in a careful, thoughtful perusal of its contents.

It is also eminently usable as a textbook for a course in the fundamental concepts of mathematics. A remarkable variety of exercises is provided, ranging in difficulty from oral examples to problems suitable for term papers. Not all the exercises have a close connection with the material developed in the text. Solutions and suggestions for solutions are given for many of the exercises.

The scope of the work is indicated by the nine chapter headings: Mathematics before Euclid, Euclid's "Elements," Non-Euclidean Geometry, Hilbert's "Grundlagen," Algebraic Structure, The Modern Mathematical Method, The Real Number System, Sets, Logic and Philosophy. There are three appendices: Euclidean Constructions, A Constructive Proof of the Existence of Transcendental Numbers, and The Axiom of Choice.

The typography of the book is excellent, and the reviewer found no misprints.

Professor Eves and Newsom are to be congratulated on producing a work of such great value for college students and their instructors.

—RALPH A. BEAVER
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College for Teachers at Albany