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Springfield, MO 65802
toshd@evangel.edu

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Magic Squares, Part 2

Lindsey Moyer, *student*

PA Epsilon

Kutztown University
Kutztown, PA 19530

Abstract

Magic squares originated over 3,000 years ago from a legend about a Divine Turtle found in the ancient Chinese book *Yih King*. According to the legend, it is mathematically possible to form $n \times n$ squares where each row, column, and diagonal add up to the same magic constant or value using the numbers 1 through n^2. Over time, mathematicians have developed different methods to compose extraordinary magic squares, other than the original Lo-Shu 3×3 magic square. My research goes in-depth with the composition of two specialized types of magic squares which are are concentric magic squares, where the outer shell of the magic square can be removed but the inner square is still magic; and pandiagonal magic squares, where the wrap around diagonals also equal the magic constant.

Contents

1. Introduction

Magic squares originated over 3,000 years ago from the legend of the Divine Turtle found in an Ancient Chinese book *Yih King*. According to the legend, each year the king made sacrifices to the river god for the

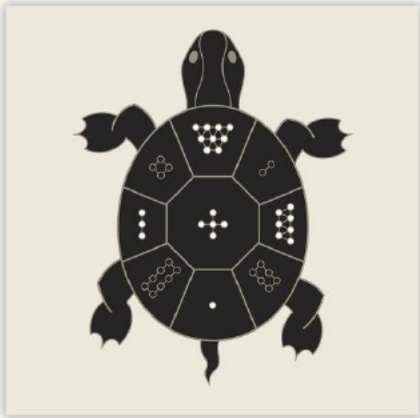

Figure 1: (Thoth, 2018)

flooded river and each year, they continued to see a turtle appear during the sacrifice. Then one year, the people noticed the turtle had dots on its shell. Looking at Figure 1, the people realized that each row, column, and diagonal added up to 15. This was the number of days in each of the 24 cycles of the Chinese Solar year. From then on, the king gave 15 sacrifices per year thanks to the Divine Turtle. Ever since that day, mathematicians have been analyzing and developing magic squares. [4][2]

One familiar creator of magic squares is Benjamin Franklin. Franklin is known for creating two unique magic squares, the 8×8 and 16×16, with the most notable being the 16×16. The 16×16 was created during a single night in the year 1767. Franklin was the clerk at the Pennsylvania Assembly meetings, and he grew very bored during these times. His friend, Mr. Logan, gave him a book on magic squares to help him pass the time. During one tedious meeting, Franklin generated his unique 16×16 magic square. He then sent this to his friends stating how magic squares were his new way to pass time. However, it was discovered years after Franklin published his magic squares that he failed to have the sums of the diagonals equal the sums of the rows and sums of the columns, so technically he did not create true magic squares. [3]

1.1 Magic Square Description

This research depends on understanding what a magic square is. A magic square is an $n \times n$ square whose rows, columns, and diagonals all add up to the same value. This research only involved normal magic squares, which are a type of magic square that only contains the integers 1 to n^2 The variable n represents the number of rows and columns in the magic square. Referring to Table 1, known as the Lo-Shu magic square, one can see this is a 3×3, since there are 3 rows and 3 columns in the square. Adding up each row, column, and diagonal yields a sum of 15. This sum of each row, column, and diagonal will be referred to as the *magic constant*. This number is not generated randomly; rather, the formula for calculating the magic constant is $\frac{n(n^2+1)}{2}$. For example, when finding the magic constant for the Lo-Shu magic square, the first step is determining what the value of n is going to be in our equation. Since we are dealing with a 3×3, the value for n is going to be 3. Plugging this into our equation gives $\frac{3(3^2+1)}{2} = \frac{3(10)}{2} = 15$. For the rest of this study, we will find the magic constant using the formula in this manner.

8	1	6
3	5	7
4	9	2

Table 1: Lo-Shu Magic Square

2. Composition

The composition of a magic square describes how a normal magic square is created or established. Every individual box inside the magic square will be referred to as a cell. The rows will be numbered from top to bottom, and the columns will be numbered from left to right. Each one is made up with a particular set of steps to produce a unique type of magic square. The two special magic squares that I researched are called concentric magic squares and pandiagonal magic squares. A concentric magic square is one that when the outside border is taken away the inner square is still magic, while the pandiagonal magic square is one where the wrap around diagonals also equal the magic constant. Comprehending the composition will prepare the way for more intricate details on magic squares in the future.

2.1 Concentric (or Bordered) Magic Squares

The concept of a concentric magic square is found in *New Recreations With Magic Squares* by Benson and Jacoby. "*A Concentric (or Bordered) Magic Square* is one that will remain magic if the borders (consisting of the top and bottom rows and the right-hand and left-hand columns) are removed" (Benson & Jacoby [1] p. 26). In Table 2, if the red cells were removed, the white cells would still create a magic square. It is important to note that the new magic square will not be normal (as defined earlier) because it will not contain the numbers 1 through n^2, but every row, column, and diagonal will still sum to the same value. We can produce either odd, singly-even, or doubly-even concentric magic squares. Each type of concentric magic square will be described by showing two examples.

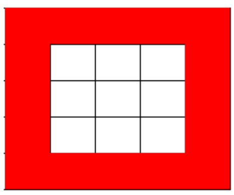

Table 2: Concentric Magic Square (Border Highlighted)

2.1.1 3×3 Magic Square

To compose any odd order concentric magic square, the following tables are essential. Table 3 is adapted from page 29 of Benson and Jacoby's book [1].

$n^2 - \frac{(n-1)}{2}$	Complements of Set B	$n^2 - \frac{3(n-1)}{2}$
Set A	Previous odd-order magic square plus $2(n\text{-}1)$	Complements of Set A
$n + \frac{(n-1)}{2}$	Set B	$1 + \frac{(n-1)}{2}$

Table 3: Odd Order Magic Square Schema

	Set A	
	n	
$n^2 - a$		$n + a$
$n^2 - b$		$n + b$
.		.
.		.
.		.
$n^2 - r$		$n + r$

Table 4: Set A

	Set B	
	n^2	
$n^2 - a'$		$n + a'$
$n^2 - b'$		$n + b'$
.		.
.		.
.		.
$n^2 - r'$		$n + r'$

Table 5: Set B

The allowable values for $a, b, \ldots, r$ and $a', b', \ldots, r'$ are from the range 1 through $n-2$. However, that range will always provide an extra value that is equal to the solution of the expression $\frac{(n-1)}{2}$ that is found in the four corners of Table 3. Therefore the solution to the expression is not an allowable value for the variables.

The first odd concentric magic square that is going to be produced is a 5×5 by going over in depth each section of Table 3. The instruction to start with is the "Previous odd-order magic square plus $2(n-1)$". The previous odd-order magic square means the odd-order magic square created using the Siamese Method of the next lowest odd size. Since we are creating a 5×5, we start with a 3×3. Thus, Table 6 shows the Lo-Shu Magic Square (from Table 1).

8	1	6
3	5	7
4	9	2

Table 6: Previous Odd-Order Magic Square (3×3)

Next, $2(n-1)$ must be added to each cell. For the entire process, the n value is the same as for the desired magic square, not the previous magic square, so $n=5$. Thus, $2(5-1)=8$ is going to be added to each cell. This step is depicted in Table 7.

16	9	14
11	13	15
12	17	10

Table 7: Previous Odd-Order Magic Square plus $2(n-1)$

When looking at Table 7, please note that this magic square is not normal because the magic constant is 39, not 15 like a normal 3×3 magic square. This is due to the fact that the numbers 1 through n^2 are not used. This instruction is now complete, so the corresponding portion can be added to the desired 5×5 as in Table 8.

	16	9	14	
	11	13	15	
	12	17	10	

Table 8: 5×5 Concentric Magic Square (Incomplete)

Thus far, the 5×5 magic square has a magic constant of 39, but the correct magic constant for a 5×5 is 65. Thus, 26 has to be added to each row, column, and diagonal so that each one adds up to 65. To accomplish this, complement pairs are going to be used. A *complement pair* is a set of two numbers that add up to a desired number. In this case, the complement pairs are going to have to add to 26. Table 8 has 16 white cells, so there are going to be 8 complement pairs.

To determine what numbers are to be used in the complement pairs, the current 5×5 is used. A normal 5×5 has to have the numbers 1 through 25, and currently the magic square already has the numbers 9 through 17, so the numbers 1 through 8 and 18 through 25 are to be used for the complement pairs. These remaining numbers are matched together to add to 26, so the complement pairs are (1,25), (2,24), (3,23), (4,22), (5,21), (6,20), (7,19), and (8,18). The complement pairs have been colored to emphasize where the complements are placed on the main magic square.

The four corner instructions in Table 3 are the next to be calculated. Solving the top left gives

$$n^2-\frac{(n-1)}{2}=5^2-\frac{(5-1)}{2}=25-2=23,$$

the top right gives

$$n^2-\frac{3(n-1)}{2}=5^2-\frac{3(5-1)}{2}=25-6=19,$$

the bottom left gives

$$n + \frac{(n-1)}{2} = 5 + \frac{(5-1)}{2} = 5 + 2 = 7,$$

and the bottom right gives

$$1 + \frac{(n-1)}{2} = 1 + \frac{(5-1)}{2} = 1 + 2 = 3.$$

Placing the corner values in the 5×5 yields the portion of the magic square found in Table 9.

23				19
	16	9	14	
	11	13	15	
	12	17	10	
7				3

Table 9: 5×5 Concentric Magic Square with Four Corner Values (Incomplete)

It is important to note that, as in Table 9, complement pairs are placed in the same diagonal. The final four sections are found simultaneously by using Tables 4 and 5. The way Set A and Set B work is that the first value in each chart is used, then we continue downward in each chart until all the values are found. For this example, Set A and Set B are going to consist of three values, which means the first value is going to be used in each set and only the first row underneath. Looking at both sets shows that the variables a and a' are going to be needed. As mentioned with the tables, the range of allowable values are going to be 1 through $n-2=3$. However, the extra value produced by the expression is $\frac{(n-1)}{2} = 2$, therefore 2 is not a possibility here, leaving 1 and 3. The variable values for a and a' get set from least to greatest so $a = 1$ and $a' = 3$.

These values are plugged into the equations in Set A and Set B, then the complements of Set A and Set B are the corresponding complement pairs. All of the calculations are shown in Table 10 and Table 11.

Set A	Complements of Set A
5	21
$5^2 - 1 = 24$	2
$5 + 1 = 6$	20

Table 10: Set A and Complements for 5×5 Concentric Magic Square

Set B	Complements of Set B
$5^2 = 25$	1
$5^2 - 3 = 22$	4
$5 + 3 = 8$	18

Table 11: Set B and Complements for 5×5 Concentric Magic Square

These portions are complete and get placed in the main magic square. Set A and its complement get placed from top to bottom in their respective sections, and Set B and its complements get placed from left to right in their respective sections. Placing these final cells concludes the 5×5 concentric magic square, which is depicted in Table 12. Look closely at the color pattern in Table 12. It depicts how each complement pair lives in the same row, column, or diagonal, thus allowing each row, column, and diagonal to sum to the magic constant of 65.

23	1	4	18	19
5	16	9	14	21
24	11	13	15	2
6	12	17	10	20
7	25	22	8	3

Table 12: 5×5 Concentric Magic Square (Complete)

It is interesting to note that the elements of Set A or Set B do not have to be placed in that exact order. For instance, the left-hand side of Set A could read from top to bottom 6, 5, and 24 as long as the complements of Set A are also altered in the same order 20, 21, and 2. Basically they can be in any order as long as the complement pair lies in either the same row, column, or diagonal. This is true for every concentric magic square.

Next, a 7×7 magic square is going to be created utilizing the same directions as in Table 3. Once again, the first instruction to start with is the "Previous odd-order magic square plus $2(n-1)$." The odd-order magic square directly before a 7×7 is a 5×5. However, it has to be the one created with the Siamese method (from Table 12), and is shown in Table 13.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Table 13: Previous Odd-Order Magic Square (5×5)

Adding $2(n-1) = 2(7-1) = 12$ to each cell yields the magic square in Table 14.

29	36	13	20	27
35	17	19	26	28
16	18	25	32	34
22	24	31	33	15
23	30	37	14	21

Table 14: Previous Odd-Order Magic Square plus $2(n-1)$

Like before, this is not a normal magic square because the magic constant here is 125, and not 65 like a normal 5×5 magic square. The completed section can now get added to the desired 7×7 as in Table 15.

	29	36	13	20	27	
	35	17	19	26	28	
	16	18	25	32	34	
	22	24	31	33	15	
	23	30	37	14	21	

Table 15: 7×7 Concentric Magic Square (Incomplete)

The expected magic constant for a 7×7 is 175, but currently each row, column, and diagonal sums to 125, so 50 has to be added to each row, column, and diagonal. Thus, the total of each complement pair will be 50 for this example. Table 15 portrays that there are 24 cells left, which allows there to be 12 complement pairs. The numbers currently in the magic square are 13 through 37, so the numbers for the complement pairs will be from the sets 1 through 12 and 38 through 49. The complement pairs are (1,49), (2,48), (3,47), (4,46), (5,45), (6,44), (7,43), (8,42), (9,41), (10,40), (11,39), and (12,38).

Next, the four corners must be found using the formulas in Table 3. The top left corner is

$$n^2 - \frac{(n-1)}{2} = 7^2 - \frac{(7-1)}{2} = 49 - 3 = 46,$$

the top right is

$$n^2 - \frac{3(n-1)}{2} = 7^2 - \frac{3(7-1)}{2} = 49 - 9 = 40,$$

the bottom left is

$$n + \frac{(n-1)}{2} = 7 + \frac{(7-1)}{2} = 7 + 3 = 10,$$

and the bottom right is

$$1 + \frac{(n-1)}{2} = 1 + \frac{(7-1)}{2} = 1 + 3 = 4.$$

Adding the four corner values to the 7×7 gives the partial magic square in Table 16. Notice how complement pairs lie on the same diagonal.

46						40
	29	36	13	20	27	
	35	17	19	26	28	
	16	18	25	32	34	
	22	24	31	33	15	
	23	30	37	14	21	
10						4

Table 16: 7×7 Concentric Magic Square with Four Corner Values (Incomplete)

The four portions left to be completed are formed using Tables 4 and 5. Looking at Table 16, it can be seen that Set A and Set B will each consist of five terms, so the first value and the next two rows are going to be used in both sets. Thus, Set A and Set B will need the variables a, a', b, and b' to be declared. The values for these come from the range 1 through $n-2=5$. As before, the value $\frac{(n-1)}{2} = \frac{6}{2} = 3$ cannot be used, leaving 1, 2, 4, and 5. Declaring these from smallest to largest yields $a=1, a'=2, b=4$, and $b'=5$. Plugging these values into the equations in Set A and Set B produce both complete sets and their complements that are found in Table 17 and Table 18.

Set A	Complements of Set A
7	43
$7^2-1=48$	2
$7+1=8$	42
$7^2-4=45$	5
$7+4=11$	39

Table 17: Set A and Complements for 7×7 Concentric Magic Square

Set B	Complements of Set B
$7^2=49$	1
$7^2-2=47$	3
$7+2=9$	41
$7^2-5=44$	6
$7+5=12$	38

Table 18: Set B and Complements for 7×7 Concentric Magic Square

These portions can now be placed in their respective areas. The elements of Set A and their complements get placed from top to bottom, while the elements of Set B and their complements get placed left to right. The completed 7×7 concentric magic square is portrayed in Table 19. Like before, each colored complement pair lies in either the same row, column, or diagonal. Furthermore, each row, column, and diagonal equals the desired magic constant of 175, so the result is a normal 7×7 magic square.

46	1	3	41	6	38	40
7	29	36	13	20	27	43
48	35	17	19	26	28	2
8	16	18	25	32	34	42
45	22	24	31	33	15	5
11	23	30	37	14	21	39
10	49	47	9	44	12	4

Table 19: 7×7 Concentric Magic Square (Complete)

As before, note that the elements of Set A or Set B do not have to be placed in that exact order. For instance, the left-hand side of Set A could read from top to bottom 45, 11, 8, 7, and 45 as long as the complements of Set A are also altered in the same order 5, 39, 42, 43, and 2. Again, they can be in any order as long as the complement pair lies in either the same row, column, or diagonal.

2.1.2 Singly-Even Concentric Magic Squares

Composing a singly-even order concentric magic square utilizes the following tables. Table 20 was adapted from page 30 of Benson and Jacoby's book [1].

n^2	Complements of Set D	n^2+1-n
Set C	Previous doubly-even magic square plus 2(n-1)	Complements of Set C
n	Set D	1

Table 20: Singly-Even Order Magic Square Schema

Set C
n^2-n-1
n^2-n-3
$n+3$
4
along with $\frac{(n-6)}{4}$ terms from each column of Set E

Table 21: Set C

Set D
n^2-1
n^2-2
n^2-n
5
along with $\frac{(n-6)}{4}$ terms from each column of Set F

Table 22: Set D

Set E			
n^2-a	$n+a$	$n^2+1-n-p$	$1+p$
n^2-b	$n+b$	$n^2+1-n-q$	$1+q$
.	.	.	.
.	.	.	.
.	.	.	.
n^2-k	$n+k$	$n^2+1-n-w$	$1+w$

Table 23: Set E

Set F			
$n^2 - a'$	$n + a'$	$n^2 + 1 - n - p'$	$1 + p'$
$n^2 - b'$	$n + b'$	$n^2 + 1 - n - q'$	$1 + q'$
.	.	.	.
.	.	.	.
.	.	.	.
$n^2 - k'$	$n + k'$	$n^2 + 1 - n - w'$	$1 + w'$

Table 24: Set F

The possible values for $a, b, \ldots, k, a', b', \ldots, k', p, q, \ldots w$, and $p', q', \ldots w'$ are from the range 5 through $n-2$. We then define the variables in order from least to greatest using the values in the range 5 through $n-2$.

A 6×6 is the first singly-even magic square that is going to be composed here. The first instruction to start with for this type is the "Previous doubly-even magic square plus $2(n-1)$," meaning the doubly-even square of size directly below the desired singly-even magic square. For this example, a 6×6 is desired, so the doubly-even square of size directly below that is the 4×4 produced with the Doubly-Even Method. The starting magic square is displayed in Table 25.

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Table 25: Doubly-Even Magic Square (4×4)

Now, $2(n-1)$ must be added to each cell. For the rest of the construction, the variable n refers to the n of the desired magic square, not the starting one. Thus, $n = 6$, so $2(6-1) = 10$ is to be added to each cell, as in Table 26.

11	25	24	14
22	16	17	19
18	20	21	15
23	13	12	26

Table 26: Previous Doubly-Even Magic Square plus $2(n-1)$

As in the previous section, this magic square is not normal because the magic constant is 74 and not 34. This portion is now completed and can be added to the 6×6 magic square, as in Table 27.

	11	25	24	14	
	22	16	17	19	
	18	20	21	15	
	23	13	12	26	

Table 27: 6×6 Concentric Magic Square (Incomplete)

The 6×6 currently has a magic constant of 74, but a normal 6×6 has a magic constant of 111, so the sum of the complement pairs is going to be 37. Adding those values to each row, column, and diagonal will produce the magic constant of 111. Table 27 yields 20 empty cells, so ten complement pairs are needed. Also, the numbers 11 through 26 are already placed in the magic square, so the numbers 1 through 10 and 27 through 36 will be the values in the complement pairs. The complement pairs will be (1,36), (2,35), (3,34), (4,33), (5,32), (6,31), (7,30), (8,29), (9,28), and (10,27).

The next portions to be completed from Table 20 are the four corners. When the formulas for these are calculated, the top left corner is $n^2 = 6^2 = 36$, the top right is $n^2 + 1 - n = 6^2 + 1 - 6 = 31$, the bottom left is $n = 6$, and the bottom right is $1 = 1$. The corner cells are now complete and can be added to the main 6×6 square such as in Table 28. Notice how complement pairs lie on the same diagonals.

36					31
	11	25	24	14	
	22	16	17	19	
	18	20	21	15	
	23	13	12	26	
6					1

Table 28: 6×6 Concentric Magic Square with Four Corner Values (Incomplete)

The last four instructions utilize Tables 21, 22, 23, and 24. The first step in completing the last four portions is to look at the formulas in Set C and Set D. The first four rows in each set get calculated along with $\frac{(n-6)}{4}$ terms from Set E and Set F. For this case $\frac{(6-6)}{4} = 0$, so Set E and Set F will not be used at all, only Set C and Set D. (A 6×6 is the only case this happens for, so the next example will utilize Set E and Set F.) There are no variables needed for Set C and Set D except for n, which is already known. Plugging n into the formulas produces the results in Table 29 and 30.

Set C	Complements of Set C
$6^2 - 6 - 1 = 29$	8
$6^2 - 6 - 3 = 27$	10
$6 + 3 = 9$	28
4	33

Table 29: Set C and Complements for 6×6 Concentric Magic Square

Set D	Complements of Set D
$6^2 - 1 = 35$	2
$6^2 - 2 = 34$	3
$6^2 - 6 = 30$	7
5	32

Table 30: Set D and Complements for 6×6 Concentric Magic Square

These values can now be plugged into their corresponding portions in the 6×6 magic square. Set C and its complements are placed from top to bottom, and Set D and its complements are placed from left to right. The final 6×6 concentric magic square is displayed in Table 31. Similar to the odd-order concentric magic squares, complements all live in the same row, column, or diagonal.

36	2	3	7	32	31
29	11	25	24	14	8
27	22	16	17	19	10
9	18	20	21	15	28
4	23	13	12	26	33
6	35	34	30	5	1

Table 31: 6×6 Concentric Magic Square (Complete)

The next singly-even concentric magic square example is going to be a 10×10. This will follow a very similar process except that terms from Set E and Set F will also be used. Starting with the "Previous doubly-even magic square plus $2(n-1)$" instruction leads to starting with the doubly-even magic square directly before a 8×8 This would be an 8×8 formed using the Doubly-Even Method, which is displayed in Table 32.

1	2	62	61	60	59	7	8
9	10	54	53	52	51	15	16
48	47	19	20	21	22	42	41
40	39	27	28	29	30	34	33
32	31	35	36	37	38	26	25
24	23	43	44	45	46	18	17
49	50	14	13	12	11	55	56
57	58	6	5	4	3	63	64

Table 32: Doubly-Even Magic Square (8×8)

Each cell now gets $2(n-1) = 2(1-1) = 18$ added to it to finish this instruction. This addition is detailed in Table 33.

19	20	80	79	78	77	25	26
27	28	72	71	70	69	33	34
66	65	37	38	39	40	60	59
58	57	45	46	47	48	52	51
50	49	53	54	55	56	44	43
42	41	61	62	63	64	36	35
67	68	32	31	30	29	73	74
75	76	24	23	22	21	81	82

Table 33: Previous Doubly-Even Magic Square plus $2(n-1)$

As in the last example, this is not a normal magic square because the magic constant is not the usual magic constant for a 8×8, but instead is 404. This portion is complete and can be added into the main 10×10 magic square as in Table 34.

	19	20	80	79	78	77	25	26	
	27	28	72	71	70	69	33	34	
	66	65	37	38	39	40	60	59	
	58	57	45	46	47	48	52	51	
	50	49	53	54	55	56	44	43	
	42	41	61	62	63	64	36	35	
	67	68	32	31	30	29	73	74	
	75	76	24	23	22	21	81	82	

Table 34: 10×10 Concentric Magic Square (Incomplete)

The current magic constant is 404; however, a true 10×10 magic square has a magic constant of 505, so 101 is going to be the sum of each complement pair, so that each row, column, and diagonal sums to the magic constant. Table 34 shows 36 empty cells, which requires there to be 18 complement pairs. The numbers 19 through 82 are already in the magic square so the complement pairs will consist of the numbers from 1 through 18 and 83 through 100. The complement pairs will be (1,100), (2,99), (3,98), (4,97), (5,96), (6,95), (7,94), (8,93), (9,92), (10,91), (11,90), (12,89), (13,88), (14,87), (15,86), (16,85), (17,84), and (18,83).

The four corners can be found by plugging $n = 10$ into each formula. After calculating, the top left corner is $n^2 = 10^2 = 100$, the top right is $n^2+1-n = 10^2+1-10 = 91$, the bottom left is $n = 10$, and the bottom right is $1 = 1$. Table 35 depicts these numbers in their appropriate corners, with complement pairs in the same diagonals.

100									91
	19	20	80	79	78	77	25	26	
	27	28	72	71	70	69	33	34	
	66	65	37	38	39	40	60	59	
	58	57	45	46	47	48	52	51	
	50	49	53	54	55	56	44	43	
	42	41	61	62	63	64	36	35	
	67	68	32	31	30	29	73	74	
	75	76	24	23	22	21	81	82	
10									1

Table 35: 10×10 Concentric Magic Square with Four Corner Values (Incomplete)

To finish the last four sections, Table 21 and Table 22 must be looked at in detail. Table 35 shows that there are eight cells left for Set C and Set D, so the first four values of each will be used. Using the formula $\frac{(n-6)}{4} = \frac{(10-6)}{4} = 1$, means that one (the first) term from each column in Set E is going to be used in Set C and the one (first term) from each column in Set F is going to be used in Set D.

Computing Set C and Set D does not involve knowing any other variable values other than n, but using the first row in Set E and Set F requires the variables a, p, a', and p'. As stated previously, the possible values for the variables are 5 through $n-2 = 8$. Defining the variables as instructed gives $a = 5, p = 6, a' = 7$, and $p' = 8$. Plugging all of these variables into Set C, Set D, the first term in each column of Set E, and the first term in each column of Set F produces the complete values for Set C and its complements, and for Set D and its complements. This is shown in Table 36 and Table 37.

Set C	Complements of Set C
$10^2 - 10 - 1 = 89$	12
$10^2 - 10 - 3 = 87$	14
$10 + 3 = 13$	88
4	97
$10^2 - 5 = 95$	6
$10 + 5 = 15$	86
$10^2 + 1 - 10 - 6 = 85$	16
$1 + 6 = 7$	94

Table 36: Set C and Complements for 10×10 Concentric Magic Square

Set D	Complements of Set D
$10^2 - 1 = 99$	2
$10^2 - 2 = 98$	3
$10^2 - 10 = 90$	11
5	96
$10^2 - 7 = 93$	8
$10 + 7 = 17$	84
$10^2 + 1 - 10 - 8 = 83$	18
$1 + 8 = 9$	92

Table 37: Set D and Complements for 10×10 Concentric Magic Square

All these may now be added to the 10×10, with the values of Set C and its complements placed top to bottom and the values of Set D and its complements placed left to right. The completed 10×10 concentric magic square is shown in Table 38. Every row, column, and diagonal adds up to the magic constant of 505 with every complement pair residing in the same row, column, or diagonal.

100	2	3	11	96	8	84	18	92	91
89	19	20	80	79	78	77	25	26	12
87	27	28	72	71	70	69	33	34	14
13	66	65	37	38	39	40	60	59	88
4	58	57	45	46	47	48	52	51	97
95	50	49	53	54	55	56	44	43	6
15	42	41	61	62	63	64	36	35	86
85	67	68	32	31	30	29	73	74	16
7	75	76	24	23	22	21	81	82	94
10	99	98	90	5	93	17	83	9	1

Table 38: 10×10 Concentric Magic Square (Complete)

2.1.3 Doubly-Even Concentric Magic Squares

Composing a doubly-even concentric magic square requires Table 39, Table 23 found in the Singly-Even Concentric Magic Squares section, and Table 40 which is an updated version of Set F used in 2.1.2. Table 39 was adapted from page 31 of the book by Benson and Jacoby [1].

$n^2 - b$	Complements of Set F	$n^2 + 1 - n - a$
Set E	Previous singly-even magic square plus 2(n-1)	Complements of Set E
$n + a$	Set F	$1 + b$

Table 39: Doubly-Even Order Magic Square Schema

Set F			
$n^2 - x$			
$n^2 + 1 - n - x$			
$n^2 - a'$	$n + a'$	$n^2 + 1 - n - p'$	$1 + p'$
$n^2 - b'$	$n + b'$	$n^2 + 1 - n - q'$	$1 + q'$
.	.	.	.
.	.	.	.
.	.	.	.
$n^2 - k'$	$n + k'$	$n^2 + 1 - n - w'$	$1 + w'$

Table 40: Set F

The possible values for the variables $a, b, \ldots, k, a', b', \ldots, k', p, q, \ldots w$, $p', q', \ldots w'$, and x are from the range 0 through $n-2$. There is one stipulation when declaring the variables and that is $2x = a + b$. The rest of the variables get defined in order.

The first doubly-even concentric magic square that will be produced here is an 8×8. The first instruction of Table 39 to start with is "Previous singly-even magic square plus $2(n-1)$." This section is similar to the "Previous doubly-even magic square plus $2(n-1)$" in creating a singly-even concentric magic square, but the roles are flipped. Last time, a doubly-even magic square was used to create a singly-even one, but now a singly-even magic square is used to create a doubly-even one. Thus, the singly-even magic square of size directly before an 8×8 is a 6×6 magic square formed with the Singly-Even Method. The starting magic square is displayed in Table 41.

35	1	6	26	19	24
3	32	7	21	23	25
31	9	2	22	27	20
8	28	33	17	10	15
30	5	34	12	14	16
4	36	29	13	18	11

Table 41: Singly-Even Magic Square (6×6)

The n value for this example or for any doubly-even concentric magic square is for the desired magic square, not the starting magic square; thus, $n = 8$. Using this value in the equation, $2(n-1)$ yields that $2(8-1) = 14$ will be added to each cell. This process is depicted in Table 42. Notice that it is not a normal 6×6 because the magic constant does not equal 111, but instead equals 195.

49	15	20	40	33	38
17	46	21	35	37	39
45	23	16	36	41	34
22	42	47	31	24	29
44	19	48	26	28	30
18	50	43	27	32	25

Table 42: Previous Singly-Even Magic Square plus $2(n-1)$

This instruction is completed and the values can be placed in the corresponding section of the current 8×8 as in Table 43.

	49	15	20	40	33	38	
	17	46	21	35	37	39	
	45	23	16	36	41	34	
	22	42	47	31	24	29	
	44	19	48	26	28	30	
	18	50	43	27	32	25	

Table 43: 8×8 Concentric Magic Square (Incomplete)

A complete 8×8 magic square should have a magic constant of 260, but it is currently 195, so the sum of each complement pair is going to be 65. Table 43 has 28 empty cells and includes numbers 15 through 50, therefore 14 complement pairs will be needed with the numbers from the ranges of 1 through 14 and 51 through 64. The complement pairs are (1,64), (2,63), (3,62), (4,61), (5,60), (6,59), (7,58), (8,57), (9,56), (10,55), (11,54), (12,53), (13,52), and (14,51).

Finding the four corner values cannot occur next, as it did in the odd and singly-even concentric cases, because the equations for the four corner values have variables that are currently unknown. To determine how many variables are used for the current problem, the number of terms used from Set E and Set F need to be calculated. As defined in Table 23, Set E includes $\frac{n}{4}$ terms per column, which would be $\frac{8}{4} = 2$ terms per column, and Set F includes the first two rows then $\frac{(n-4)}{4}$ terms per column, which would be the first two rows then $\frac{(8-4)}{4} = \frac{4}{4} = 1$ term per column. Thus, the variables a, b, p, q, a', p', and x have to be defined. We use the range mentioned earlier, so the range is 0 through $n-2 = 6$. Using the stipulation that $2x = a+b$ we chose x to be 2, which would make the left side of the equation equal 4, leaving a to be 1 and b to be 3. Defining the remaining variables in order gives $p = 0, q = 4, a' = 5$, and $p' = 6$.

The four corner values are now able to be found. The resulting corner values are: the top left equaling $n^2 - b = 8^2 - 3 = 61$, the top right equaling $n^2 + 1 - n - a = 8^2 + 1 - 8 - 1 = 56$, the bottom left equaling $n + a = 8 + 1 = 9$, and the bottom right equaling $1+b=1+3=4$. These values are now able to be plugged into the 8×8 magic square, as in Table 44.

61							56
	49	15	20	40	33	38	
	17	46	21	35	37	39	
	45	23	16	36	41	34	
	22	42	47	31	24	29	
	44	19	48	26	28	30	
	18	50	43	27	32	25	
9							4

Table 44: 8×8 Concentric Magic Square with Four Corner Values (Incomplete)

The last four portions can now be found because all the variables have already been defined. However, Set E and its complements do not behave the same as every other set has so far. Using Table 23, Set E will produce two extra terms than what is needed that are the same as the corner values. These values and their complement pair values are disregarded. Set F and its complements produces the exact number needed so nothing is to be disregarded. The completed sets are displayed in Table 45 and Table 46. Note in Table 45, the repeated corner values have an X with them.

Set E	Complements of Set E
$8^2-1=63$	2
X $[8+1=9]$ X	X [56] X
$8^2+1-8-0=57$	8
$1+0=1$	64
X $[8^2-3=61]$ X	X [4] X
$8+3=11$	54
$8^2+1-8-4=53$	12
$1+4=5$	60

Table 45: Set E and Complements for 8×8 Concentric Magic Square

Set F	Complements of Set F
$8^2-2=62$	3
$8^2+1-8-2=55$	10
$8^2-5=59$	6
$8+5=13$	52
$8^2+1-8-6=51$	14
$1+6=7$	58

Table 46: Set F and Complements for 8×8 Concentric Magic Square

These four completed sets may now be added to the 8×8 magic square. Set F and its complements can be placed as before from left to right, but Set E and its complements have to be carefully placed. They are still placed from top to bottom, however, any number that has X with it gets skipped. For instance, the order for Set E would be 63, (skip 9), 57, 1, (skip 61), 11, 53, and 5. The complete 8×8 concentric magic square is portrayed in Table 47.

61	3	10	6	52	14	58	56
63	49	15	20	40	33	38	2
57	17	46	21	35	37	39	8
1	45	23	16	36	41	34	64
11	22	42	47	31	24	29	54
53	44	19	48	26	28	30	12
5	18	50	43	27	32	25	60
9	62	55	59	13	51	7	4

Table 47: 8×8 Concentric Magic Square (Complete)

The final concentric magic square that will be composed here is a 12×12 concentric magic square. The process will be very similar, with the first section being the "Previous singly-even magic square plus $2(n-1)$." Since a 12×12 is the goal, the starting magic square is going to be the singly-even magic square of size directly below, a 10×10 magic square, composed by the Singly-Even Method. This magic square is exhibited here in Table 48.

92	99	1	8	15	67	74	51	58	40
98	80	7	14	16	73	55	57	64	41
4	81	88	20	22	54	56	63	70	47
85	87	19	21	3	60	62	69	71	28
86	93	25	2	9	61	68	75	52	34
17	24	76	83	90	42	49	26	33	65
23	5	82	89	91	48	30	32	39	66
79	6	13	95	97	29	31	38	45	72
10	12	94	96	78	35	37	44	46	53
11	18	100	77	84	36	43	50	27	59

Table 48: Singly-Even Magic Square (10×10)

In this case, $n = 12$, so $2(12-1) = 2(11) = 22$ has to be added to each cell. Adding 22 to each cell produces Table 49. Again, note that Table 49 is not a normal 10×10 because the magic constant is 725 and not 505.

114	121	23	30	37	89	96	73	80	62
120	102	29	36	38	95	77	79	86	63
26	103	110	42	44	76	78	85	92	69
107	109	41	43	25	82	84	91	93	50
108	115	47	24	31	83	90	97	74	56
39	46	98	105	112	64	71	48	55	87
45	27	104	111	113	70	52	54	61	88
101	28	35	117	119	51	53	60	67	94
32	34	116	118	100	57	59	66	68	75
33	40	122	99	106	58	65	72	49	81

Table 49: Previous Singly-Even Magic Square plus $2(n-1)$

This portion is complete and may be added to the desired 12×12 magic square as in Table 50.

	114	121	23	30	37	89	96	73	80	62	
	120	102	29	36	38	95	77	79	86	63	
	26	103	110	42	44	76	78	85	92	69	
	107	109	41	43	25	82	84	91	93	50	
	108	115	47	24	31	83	90	97	74	56	
	39	46	98	105	112	64	71	48	55	87	
	45	27	104	111	113	70	52	54	61	88	
	101	28	35	117	119	51	53	60	67	94	
	32	34	116	118	100	57	59	66	68	75	
	33	40	122	99	106	58	65	72	49	81	

Table 50: 12×12 Concentric Magic Square (Incomplete)

The current rows, columns, and diagonals of the 12×12 magic square add up to 725, but a normal 12×12 magic square has a magic constant of 870, so the value of each complement pair is going to be 145. Table 50 has 44 empty cells, thus, 22 complement pairs will be needed. Currently, numbers 23 through 122 are in the magic square, so the complement pair numbers will be from the ranges of 1 through 22 and 123 through 144. Therefore, the complement pairs will be (1,144), (2,143), (3,142), (4,141), (5,140), (6,139), (7,138), (8,137), (9,136), (10,135), (11,134),(12,133), (13,132), (14,131), (15,130), (16,129), (17,128), (18,127), (19,126), (20,125), (21,124), and (22,123).

Before the corner values can be found, the values for the variables must be declared. First, the number of variables must be found by looking at Set E in Table 23 from the last section and Set F in Table 40. Using the n value of 12 in the expression $\frac{n}{4}$, there are $\frac{12}{4} = 3$ terms per column to be used from Set E, and $\frac{(n-4)}{4} = \frac{(1-4)}{4} = \frac{8}{4} = 2$ terms per column from Set F along with the first two terms of Set F. Thus, the variables that need to be defined are $a, b, c, p, q, r, a', p', b', q'$, and x. The range of values to be used is 0 through $n-2 = 12-2 = 10$. The condition that must be followed is that $2x = a + b$. A choice of $x = 2$ makes the left side of the equation 4, causing $a = 1$ and $b = 3$. This concurs with the condition since 2(2) = 4 = 1 + 3. Declaring the rest of the variables in order yields $c = 0, p = 4, q = 5, r = 6, a' = 7, p' = 8, b' = 9$, and $q' = 10$.

The four corner entries may now be calculated using Table 39 and the values of the variables just declared. After calculation, the top left corner is $n^2 - b = 12^2 - 3 = 141$, the top right corner is $n^2 + 1 - n - a = 12^2 + 1 - 12 - 1 = 132$, the bottom left corner is $n + a = 12 + 1 = 13$, and the bottom right corner is $1 + b = 1 + 3 = 4$. These 4 entries may now be added to the current 12×12 magic square as shown in Table 51.

141											132
	114	121	23	30	37	89	96	73	80	62	
	120	102	29	36	38	95	77	79	86	63	
	26	103	110	42	44	76	78	85	92	69	
	107	109	41	43	25	82	84	91	93	50	
	108	115	47	24	31	83	90	97	74	56	
	39	46	98	105	112	64	71	48	55	87	
	45	27	104	111	113	70	52	54	61	88	
	101	28	35	117	119	51	53	60	67	94	
	32	34	116	118	100	57	59	66	68	75	
	33	40	122	99	106	58	65	72	49	81	
13											4

Table 51: 12×12 Concentric Magic Square with Four Corner Values (Incomplete)

Set E and its complements and Set F and its complement are the last four portions to calculate. Since all the variables have been defined, the four sections can be determined as displayed in Table 52 and Table 53. Similar to the last example, Set E and its complements will have two extra terms in it that are the same values as the corner cells, so those terms have an X placed next to them and will be skipped over when placing the numbers in the magic square.

Set E	Complements of Set E
$12^2 - 1 = 143$	2
X $[12 + 1 = 13]$ X	X [132] X
$12^2 + 1 - 12 - 4 = 129$	16
$1 + 4 = 5$	140
X $[12^2 - 3 = 141]$ X	X [4] X
$12 + 3 = 15$	130
$12^2 + 1 - 12 - 5 = 128$	17
$1 + 5 = 6$	139
$12^2 - 0 = 144$	1
$12 + 0 = 12$	133
$12^2 + 1 - 12 - 6 = 127$	18
$1 + 6 = 7$	138

Table 52: Set E and Complements for 12×12 Concentric Magic Square

Set F	Complements of Set F
$12^2 - 2 = 142$	3
$12^2 + 1 - 12 - 2 = 131$	14
$12^2 - 7 = 137$	8
$12 + 7 = 19$	126
$12^2 + 1 - 12 - 8 = 125$	20
$1 + 8 = 9$	136
$12^2 - 9 = 135$	10
$12 + 9 = 21$	124
$12^2 + 1 - 12 - 10 = 123$	22
$1 + 10 = 11$	134

Table 53: Set F and Complements for 12×12 Concentric Magic Square

These four portions may now be added to the 12×12 magic square. When adding Set E and its complements, place the numbers from top to bottom, making sure to skip the X terms and placing Set F and its complements from left to right. These placements are shown in Table 54. This is now the completed 12×12 concentric magic square. As with all of the examples, the pattern can be seen that all complement pairs lie on the same row, column, and diagonal.

141	3	14	8	126	20	136	10	124	22	134	132
143	114	121	23	30	37	89	96	73	80	62	2
129	120	102	29	36	38	95	77	79	86	63	16
5	26	103	110	42	44	76	78	85	92	69	140
15	107	109	41	43	25	82	84	91	93	50	130
128	108	115	47	24	31	83	90	97	74	56	17
6	39	46	98	105	112	64	71	48	55	87	139
144	45	27	104	111	113	70	52	54	61	88	1
12	101	28	35	117	119	51	53	60	67	94	133
127	32	34	116	118	100	57	59	66	68	75	18
7	33	40	122	99	106	58	65	72	49	81	138
13	142	131	137	19	125	9	135	21	123	11	4

Table 54: 12×12 Concentric Magic Square (Complete)

2.2 Doubly-Even Pandiagonal Magic Squares

A *Pandiagonal Magic Square* has the same properties of a magic square with the additional property that the diagonals in both directions that wrap around the edge of the square also sum to the magic constant. The idea for a pandiagonal magic square is shown in Table 55. In this example, every row, column, and diagonal sum to the magic constant but also all of the blue cells, pink cells, green cells, and orange cells. Similar to the Concentric Magic Square, Benson and Jacoby [1] established a process to compose doubly-even pandiagonal magic squares that involves a generating square, a primary square formed with five requirements, and four additional steps.

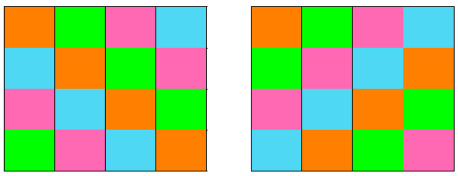

Table 55: Pandiagonal Magic Square Diagonal Patterns

There are two general items to cover before creating the magic squares that will be used in both the 4×4 and 8×8 examples presented below. The generating square that will be used in every doubly-even pandiagonal magic square created is depicted in Table 56 and found on page 71 of Benson and Jacoby's book. The pandiagonal magic square will always be composed with four $\frac{n}{2} \times \frac{n}{2}$ subsquares.

+0,+0	-0,+0
+0,-0	-0,-0

Table 56: Generating Square for a Pandiagonal Magic Square

2.2.1 4×4 Pandiagonal Magic Square

The first step in constructing a 4×4 pandiagonal magic square is to determine the common dimensions of each subsquare. Using the formula mentioned above, $\frac{4}{2} \times \frac{4}{2} = 2 \times 2$; thus, each subsquare is going to be a 2×2. Each of the subsquares will eventually be formed from a certain primary square, but first this primary square must be determined. I consider this the hardest part of the composition. The primary square is going to be of the same dimensions as each subsquare and is going to be set up like Table 57, where "v" represents the v-values, "w" represents the w-values, and "v, w" represents "$vn + w$", where n is the order of the magic square.

v,w	v,w
v,w	v,w

Table 57: 2×2 Primary Square for the 4×4 Pandiagonal Magic Square

The following list are the five requirements for creating the primary square found on pages 74 and 75 of the book by Benson and Jacoby [1].

- **Requirement A:** There are $\frac{n^2}{4}$ possible pairs of unsigned integers from 0 to $\frac{n}{2} - 1$. Once $+$ and $-$ signs are placed on each of these integers, each of the n^2 pairs of $\pm 0, \pm 1, \ldots, \pm\left(\frac{n}{2} - 1\right)$ for v, with $\pm 0, \pm 1, \ldots, \pm\left(\frac{n}{2} - 1\right)$ for w must appear once, and only once.
  - **Example:** If $+3, -1$ appears in the primary square, then $+3, +1$, $-3, -1$, and $-3, +1$ cannot appear in the primary square.
- **Requirement B:** Each row must contain $\frac{n}{4}$ positive and $\frac{n}{4}$ negative w-values.
- **Requirement C:** Each column must contain $\frac{n}{4}$ positive and $\frac{n}{4}$ negative v-values.
- **Requirement D:** The sum of the w-values in each row must equal zero.
- **Requirement E:** The sum of the v-values in each column must equal zero.

To develop the primary square, each requirement will be considered in turn. Starting with Requirement A, it seems that the first step to accomplish is determining how many possible pairs of unsigned integers there will be. Plugging $n = 4$ into the expression $\frac{n^2}{4}$ yields $\frac{4^2}{4} = 4$ possible pairs. To determine what numbers are going to be used in the v and w places, the expression $\frac{n}{2} - 1$ represents the highest value to be used. The resulting number is $\frac{4}{2} - 1 = 1$, so the numbers 0 and 1 will be used in the v and w places. Therefore, the four combinations will be (0,0), (0,1), (1,0), and (1,1).

Now that the four pairs are known, the next issue to decide on is which cell in the primary square to place each pair. Consider Table 58, in which one possibility is given for the primary square, the same number is in the v place in each column and the same number is in the w place in each row, which will satisfy Requirements D and E once the positive and negative signs are added. Next, we place the $\pm$ with each v and w place. Consider Requirements D and E at the same time. Each requirement states that the sum of the two numbers must be zero concerning the w's in each row and the v's in each column. The only way to sum to zero with these two numbers is to have $+0$ with -0 and $+1$ with -1 because $+0-0=0$ and $+1-1=0$. This means that the w's in each row and the v's in each column must be either both 0 or both 1.

1,1	0,1
1,0	0,0

Table 58: 2×2 Primary Square: Unsigned Pairs for the 4×4 Pandiagonal Magic Square

Considering Requirements B and C, the placement of the positive and negative signs must contain $\frac{n}{4}$ positive and negative signs in the w spot in each row and in the v spot in each column. To be able to visualize the positive and negative signs more clearly, the positive signs will be colored blue $(+)$ and the negative signs will be colored pink $(-)$. Using the n value of 4, Requirements B and C indicate that $\frac{4}{4}=1$ positive and negative sign has to be in the w spot in each row and in the v spot in each column. To start placing the signs, let's add a $+$ to both 1's in the top left cell. This means that the other 1 in the bottom left corner of the same column would have to be the opposite sign $-$. This also means that the 0 and the 1 in the top right corner must each have the opposite sign $-$. Now that the 0 in the top right is $-$, the 0 in the v place in the bottom right corner must be the opposite sign, $+$. Similarly, the bottom left 0 must be $-$ and the bottom right 0 must be $+$. All of the signs are placed in Table 59.

+1,+1	-0,-1
-1,-0	+0,+0

Table 59: 2×2 Primary Square: Requirements B, C, D, and E for the 4×4 Pandiagonal Magic Square

It is very important to realize here that the primary square is not unique. This process could have been started by making both 1's $-$ instead of $+$. We must always double-check to make sure all the requirements are satisfied: each combination is used, the v's in each column add up to zero, the w's in each row add up to zero, each row contains one positive and negative w, and each column contains one positive and negative v.

The four steps can now be followed to create the pandiagonal magic square and are from page 75 of the book by Benson and Jacoby [1].

Step 1: Use the generating square to place variants of the primary square into the pandiagonal square. The desired magic square is a 4×4. The primary square is a 2×2, so a 2×2 square is going to be placed in each quarter of the pandiagonal square according to the instructions implied by the generating square. If the generating square has a $+$ in a v or w place, then the v or w place of the primary square is copied into the pandiagonal magic square as is, but if the generating square has a $-$ in a v or w place, then the v or w place of the primary square gets switched to the opposite sign of what the sign in the primary square is.

Always start with the top left cell of the generating square as in Table 60, and the top left of the pandiagonal magic square. Since the sign of the v and w place is $+$ in the generating square, the primary square just gets copied over into the pandiagonal magic square as in Table 61.

+0,+0	-0,+0
+0,-0	-0,-0

Table 60: Generating Square (from Table 56): Top Left Corner Highlighted

+1,+1	-0,-1		
-1,-0	+0,+0		

Table 61: Pandiagonal 4×4 Magic Square: Top Left Corner (only) (Preliminary Step)

Now focusing on the top right corner of the generating square, the sign of the w place is $+$ again, so each w place in the primary square can be copied over to the pandiagonal magic square as is. However, the sign of the v place in the generating square is $-$, so every v sign in the primary square must be switched to the opposite sign before writing it in the pandiagonal magic square. The current part of the generating square is highlighted in Table 62, and the pandiagonal magic square is updated in Table 63.

+0,+0	-0,+0
+0,-0	-0,-0

Table 62: Generating Square (from Table 56): Top Right Corner Highlighted

+1,+1	-0,-1	-1,+1	+0,-1
-1,-0	+0,+0	+1,-0	-0,+0

Table 63: Pandiagonal 4×4 Magic Square: Top Left and Right Corners (only) (Preliminary Step)

The bottom left corner is handled similar to the top right corner except here the $-$ lies in the w place and the $+$ lies in the v place. This means that the v place of the primary square can be copied over, but the w place of the primary square must be switched before writing the values in the pandiagonal magic square. The current section of the generating square and the updated pandiagonal magic square are depicted in Table 64 and Table 65.

+0,+0	-0,+0
+0,-0	-0,-0

Table 64: Generating Square (from Table 56): Bottom Left Corner Highlighted

+1,+1	-0,-1	-1,+1	+0,-1
-1,-0	+0,+0	+1,-0	-0,+0
+1,-1	-0,+1		
-1,+0	+0,-0		

Table 65: Pandiagonal 4×4 Magic Square: Top Left, Top Right, and Bottom Left Corners (only) (Preliminary Step)

The generating square in the bottom left corner is both $-$, so both the v and w place of the primary square must be switched before placing them in the pandiagonal magic square. Both the generating square and pandiagonal magic square bottom right corner are highlighted and completed in Table 66 and 67.

+0,+0	-0,+0
+0,-0	-0,-0

Table 66: Generating Square (from Table 56): Bottom Right Corner Highlighted

+1,+1	-0,-1	-1,+1	+0,-1
-1,-0	+0,+0	+1,-0	-0,+0
+1,-1	-0,+1	-1,-1	+0,+1
-1,+0	+0,-0	+1,+0	-0,-0

Table 67: Pandiagonal 4×4 Magic Square: Step 1 Completed (Preliminary Step)

Step 2: Add $n-1$ to each negative number and then remove all $+,-$ signs. In this case, $4-1=3$, so 3 is going to be added to each negative number. The two possible cases here are either $-0+3=3$ or $-1+3=2$. Completing Step 2 produces the square in Table 68.

1,1	3,2	2,1	0,2
2,3	0,0	1,3	3,0
1,2	3,1	2,2	0,1
2,0	0,3	1,0	3,3

Table 68: Pandiagonal 4×4 Magic Square: Step 2: $(n-1)$ Added to Each Negative Number (Preliminary Step)

Step 3: Plug the v value and w value into the vn+w formula for each cell in turn. To show an example of what each cell would look like, consider the first cell in the top row. The values to plug in are $(v,w) = (1,1)$, so the calculations would be 1(4) + 1 = 5. Thus, 5 would now be the number that gets placed in that same cell. All of the calculation results are displayed in Table 69.

5	14	9	2
11	0	7	12
6	13	10	1
8	3	4	15

Table 69: Pandiagonal 4×4 Magic Square: Step 3: Calculation Results using $vn+w$ (Preliminary Step)

Notice in Table 69 that one of the cells contains the number 0. That is not a number that is allowed to be in a normal magic square.

Step 4: Add 1 to each cell. This step is finalized in Table 70.

6	15	10	3
12	1	8	13
7	14	11	2
9	4	5	16

Table 70: Pandiagonal 4×4 Magic Square: Step 4: Adding 1 to Each Cell (Complete)

To prove that this is a complete 4×4 pandiagonal magic square, the cells have been colored to depict the wrap around diagonals. These diagonal values should be added up to verify that they equal the magic constant of 34. This whole process is shown in Table 71.

6	15	10	3
12	1	8	13
7	14	11	2
9	4	5	16

6+1+11+16 = 34
3+12+14+5 = 34
10+13+7+4 = 34
15+8+2+9 = 34

6	15	10	3
12	1	8	13
7	14	11	2
9	4	5	16

6+13+11+4 = 34
3+8+14+9 = 34
10+1+7+16 = 34
15+12+2+5 = 34

Table 71: Complete 4×4 Pandiagonal Magic Square

2.2.2 8 × 8 Pandiagonal Magic Square

The pandiagonal magic square of order 8 follows the same process as the 4×4, except some parts can tend to get a little more confusing because more numbers are used than just 0 and 1 this time. The dimensions of the primary square are going to be $\frac{n}{2}\times\frac{n}{2}=\frac{8}{2}\times\frac{8}{2}=4\times4$. Similar to the 4×4 case, we begin by filling in the primary square using the requirements.

Using the formulas from Requirement A, the calculations show that there will be $\frac{n^2}{4}=\frac{8^2}{4}=16$ possible pairs of unsigned numbers 0 through $\frac{n}{2}-1=\frac{8}{2}-1=3$. Thus these possible pairs will be (0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3), (2,0), (2,1), (2,2), (2,3), (3,0), (3,1), (3,2), and (3,3). One possibility for the primary square is shown in Table 72.

1,1	0,2	2,0	3,3
1,2	0,1	2,3	3,0
1,3	0,0	2,1	3,2
1,0	0,3	2,2	3,1

Table 72: 4 × 4 Primary Square for 8 × 8 Pandiagonal Magic Square: Unsigned Pairs

To determine why we placed these pairs as we did in the primary square, consider Requirements D and E. Using the numbers 0, 1, 2, and 3, there are a couple of ways to get a signed row or column to equal 0 once we add signs to the numbers. One way is to have identical numbers in the same place in either a row or column and have two of them $+$ and two of them $-$, or having 0 and 3 be the same sign and 1 and 2 be the same sign. Both of these approaches are going to be used in this primary square. In Table 72, the v's in the columns are all the same number and the w's in the rows will add up to zero by having the 0 and 3 be the same sign and 1 and 2 be the same sign.

Now Requirements B and C must be considered and this means that $\frac{n}{4}=\frac{8}{4}=2$ positive and negative signs must be placed in the w's in each row and in the v's in each column. Starting with the v's in the columns, the 1^{st} column of v's (1's) will alternate $+$ and $-$ downwards as will the 3^{rd} column of v's (2's). The 2^{nd} column of v's (0's) will alternate $-$ and $+$ downwards as will the 4^{th} column of v's (3's). The same pattern for the v's of the 1^{st} column is going to be used on the w's in the 1^{st} column. Since the w's in each row have to equal zero and have to have two positive and negative signs, the 2 in the top row would have to also be $+$ and the 0 and the 3 would have to be $-$. This would produce $+1+2-0-3=0$, which works for all Requirements. That same type of thinking is repeated for the rest of the rows. All of the signs are pictured in Table 73.

+1,+1	-0,+2	+2,-0	-3,-3
-1,-2	+0,-1	-2,+3	+3,+0
+1,+3	-0,+0	+2,-1	-3,-2
-1,-0	+0,-3	-2,+2	+3,+1

Table 73: 4×4 Primary Square: Requirements B, C, D, and E for 8×8 Pandiagonal Magic Square

The Requirements are complete and the primary square is created. Similar to the last example, there are many possible primary squares that could be created and this is just one of them. The four steps must now be followed in order to compose the magic square.

Step 1: Follow the generating square which is shown in Table 56. Here, instead of showing each corner of the magic square being placed one at a time, the entire magic square is placed following the generating square. The top left corner is identical to the primary square. The bottom right corner has every sign switched to the opposite of the primary square. The top right corner contains all the same w's place signs as the primary square but opposite signs in the v's place. The bottom left corner contains all the same v's place signs as the primary square but the opposite signs in the w's place. Step 1 is portrayed in Table 74.

+1,+1	-0,+2	+2,-0	-3,-3	-1,+1	+0,+2	-2,-0	+3,-3
-1,-2	+0,-1	-2,+3	+3,+0	+1,-2	-0,-1	+2,+3	-3,+0
+1,+3	-0,+0	+2,-1	-3,-2	-1,+3	+0,+0	-2,-1	+3,-2
-1,-0	+0,-3	-2,+2	+3,+1	+1,-0	-0,-3	+2,+2	-3,+1
+1,-1	-0,-2	+2,+0	-3,+3	-1,-1	+0,-2	-2,+0	+3,+3
-1,+2	+0,+1	-2,-3	+3,-0	+1,+2	-0,+1	+2,-3	-3,-0
+1,-3	-0,-0	+2,+1	-3,+2	-1,-3	+0,-0	-2,+1	+3,+2
-1,+0	+0,+3	-2,-2	+3,-1	+1,+0	-0,+3	+2,-2	-3,-1

Table 74: 8×8 Pandiagonal Magic Square: Step 1 Completed (Preliminary Step)

Step 2: Add $(n-1) = (8-1) = 7$ to each negative number. In this example the four possible cases are $-0+7 = 7$,-1 + 7 = 6, $-2+7 = 5$, and $-3+7 = 4$. Completing Step 2 creates the numbers found in Table 75.

1,1	7,2	2,7	4,4	6,1	0,2	5,7	3,4
6,5	0,6	5,3	3,0	1,5	7,6	2,3	4,0
1,3	7,0	2,6	4,5	6,3	0,0	5,6	3,5
6,7	0,4	5,2	3,1	1,7	7,4	2,2	4,1
1,6	7,5	2,0	4,3	6,6	0,5	5,0	3,3
6,2	0,1	5,4	3,7	1,2	7,1	2,4	4,7
1,4	7,7	2,1	4,2	6,4	0,7	5,1	3,2
6,0	0,3	5,5	3,6	1,0	7,3	2,5	4,6

Table 75: 8×8 Pandiagonal Magic Square: Step 2: $(n-1)$ Added to Each Negative Number (Preliminary Step)

Step 3: Apply the formula $vn + w$. An example to illustrate this would be taking the last row, last cell into account. This would become 4(8) + 6 = 32 + 6 = 38, which would be placed in this cell. All of the calculations are depicted in Table 76.

9	58	23	36	49	2	47	28
53	6	43	24	13	62	19	32
11	56	22	37	51	0	46	29
55	4	42	25	15	60	18	33
14	61	16	35	54	5	40	27
50	1	44	31	10	57	20	39
12	63	17	34	52	7	41	26
48	3	45	30	8	59	21	38

Table 76: 8×8 Pandiagonal Magic Square: Step 3: Calculation Results using $vn + w$ (Preliminary Step)

Looking at Table 76, it can be seen that one of the cells contains the number 0, which is not a part of a normal magic square.

Step 4: Add 1 to each cell. The new numbers in each cell are shown in Table 77.

10	59	24	37	50	3	48	29
54	7	44	25	14	63	20	33
12	57	23	38	52	1	47	30
56	5	43	26	16	61	19	34
15	62	17	36	55	6	41	28
51	2	45	32	11	58	21	40
13	64	18	35	53	8	42	27
49	4	46	31	9	60	22	39

Table 77: 8×8 Pandiagonal Magic Square: Step 4: Adding 1 to Each Cell (Complete)

To check to see that this is a complete 8×8 pandiagonal magic square, the cells have been colored to highlight the wrap around diagonal. All of the cells in each highlighted diagonal should add up to 260 as well as each row, column, and diagonal. Tables 78a and 78b adds each highlighted wrap around diagonal together to prove that they all have the same sum.

10	59	24	37	50	3	48	29	10+7+23+26+55+58+42+39 = 260
54	7	44	25	14	63	20	33	29+54+57+43+36+11+8+22 = 260
12	57	23	38	52	1	47	30	48+33+12+5+17+32+53+60 = 260
56	5	43	26	16	61	19	34	3+20+30+56+62+45+35+9 = 260
15	62	17	36	55	6	41	28	50+63+47+34+15+2+18+31 = 260
51	2	45	32	11	58	21	40	37+14+1+19+28+51+64+46 = 260
13	64	18	35	53	8	42	27	24+25+52+61+41+40+13+4 = 260
49	4	46	31	9	60	22	39	59+44+38+16+6+21+27+49 = 260

Table 78a: Complete 8×8 Pandiagonal Magic Square

10	59	24	37	50	3	48	29	10+33+47+61+55+32+18+4 = 260
54	7	44	25	14	63	20	33	29+20+1+16+36+45+64+49 = 260
12	57	23	38	52	1	47	30	48+63+52+26+17+2+13+39 = 260
56	5	43	26	16	61	19	34	3+14+38+43+62+51+27+22 = 260
15	62	17	36	55	6	41	28	50+25+23+5+15+40+42+60 = 260
51	2	45	32	11	58	21	40	37+44+57+56+28+21+8+9 = 260
13	64	18	35	53	8	42	27	24+7+12+34+41+58+53+31 = 260
49	4	46	31	9	60	22	39	59+54+30+19+6+11+35+46 = 260

Table 78b: Complete 8×8 Pandiagonal Magic Square

3. Conclusion

The research conducted on magic squares proved that a simple concept of an $n \times n$ square having each row, column, and diagonal equaling the same value can be very complex when studied closely. Examining concentric magic squares provides a very interesting technique on how to create a magic square when the inner square is magic as well as the full square. Understanding how to create pandiagonal magic squares yields fascinating patterns where each wrap around diagonal also equals the magic constant. It is intriguing that over 3,000 years ago a Divine Turtle brought a mystery to this world that is still being studied today and for years to come.

References

[1] Benson,W. H., & Jacoby, O. (1976). *New recreations with magic squares.* Dover Publications.

[2] Hurley, T. (n.d.). *The fascinating history of magic squares*. lovetoknow. `https://feng-shui.lovetoknow.com/history-magic-squares`

[3] Kolpas, S. J. (2019). *Mathematical treasure: Benjamin Franklin's magic square of squares. Convergence.* https://www.maa.org/press/periodicals/convergence/mathematical-treasure-benjamin-franklin-s-magic-square-of-squares#:\~:text=Franklin's\%20Magic\%20Square\%20of\%20Squares.,vertical\%20or\%20horizontal\%2C\%20is\%202056

[4] Thoth, A. (2018). Divine Turtle [Photograph]. *The spruce*. https://www.thespruce.com/feng-shui-magic-of-the-lo-shu-square-1274879

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Editor's Note

Lindsey Moyer's work won third place for presentations at the (virtual) 43^{rd} Biennial Convention hosted by the University of Central Missouri in Warrensburg, Missouri, April 15-17, 2021. This is the second of two parts, the first appeared in the Spring 2021 issue of *The Pentagon*.

Non-Realizable Polynomial Root Sequences

Laura J. Batts, *student*
Megan E. Moran, *student*

IN Gamma

Anderson University
Anderson, IN 46012

Abstract

Rolle's theorem is a classical result typically studied in first-semester Calculus that can be used to locate the roots of a derivative of a polynomial between two consecutive roots of the polynomial. We investigate polynomials with distinct real roots, whose derivatives also have distinct real roots, none of which coincide. By differentiating several times, we produce a sequence of the roots of all nontrivial derivatives of a polynomial. For a polynomial of degree n this process produces $n(n+1)/2$ distinct roots. Although the ordering of the roots is constrained by Rolle's theorem, surprisingly not all root sequences allowed by Rolle's theorem exist. We investigate these non-realizable root sequences and establish elementary proofs for the non-realizability of particular root sequences.

Introduction

Our setting for this work is to examine polynomials with all distinct real roots, and whose non-trivial derivatives also have distinct real roots. Rolle's theorem dictates the orderings of some of these roots, however there is more subtlety to their orderings.

A *root sequence* is an ordering of all the roots of a polynomial and the roots of all of that polynomial's non-trivial derivatives. Assuming that the roots of our polynomial are all real and all are distinct from one another, a root sequence is an ordering of $n(n+1)/2$ real numbers. Our motivating question asks, "What root sequences are possible from polynomials with all distinct real roots?"

As we do not care about the particular values of the roots, only how they are ordered in relationship to one another, the following notation is helpful.

The notation we use for polynomial P of degree n is as follows:

- Use 0's to denote the n roots of P
- Use 1's to denote the $n-1$ roots of P'
- Use 2's to denote the $n-2$ roots of P''

$\vdots$

- Use k's to denote the $n-k$ roots of $P^{(k)}$

  $\vdots$

- Use $n-1$ to denote the 1 root of $P^{(n-1)}$

Something interesting to note is that if these roots are all distinct, we have sequence of length $1+2+3+\ldots+n = n(n+1)/2$.

Rolle's Theorem and Root Sequences

Rolle's theorem constrains the ordering of the roots in our root sequences. This is demonstrated in Figure 1:

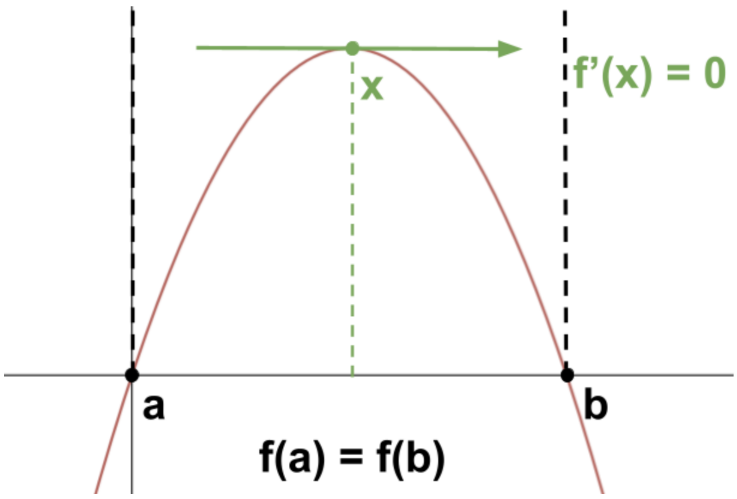

Figure 1: For the polynomial P, a root of P' must occur between any two distinct roots of P.

Translated into the setting of our root sequences, Rolle's theorem implies that for every pair of consecutive k's in our root sequences, there must occur a $k+1$.

Root Sequence Examples

We examine several examples of root sequences to show what types of patterns emerge in their structures.

Degree 1

The first root sequence that we are able to construct is that of a root sequence of 0. This sequence represent the zero of a linear equation or where the line crosses the x-axis. In the figure below, the red line represents this zero. With a linear equation we do not have any non-trivial derivatives that produce any more roots so this is the only part of the sequence and our degree here is $n = 1$.

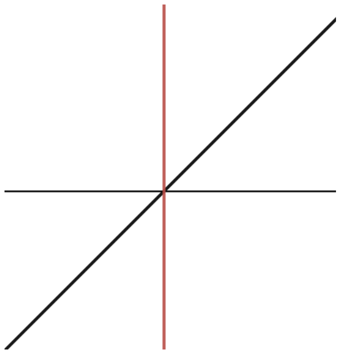

Figure 2: Linear polynomials have a root sequence of 0.

Degree 2

In the next graph, we move up to a quadratic of degree $n = 2$ and we increase the number and type of roots we can have in our root sequences. The red lines represent the zeroes of the quadratic and the orange dotted line represents the root of the first derivative of the quadratic. By Rolle's theorem, we know that the 1 must go between the two 0's here and our sequence becomes 010.

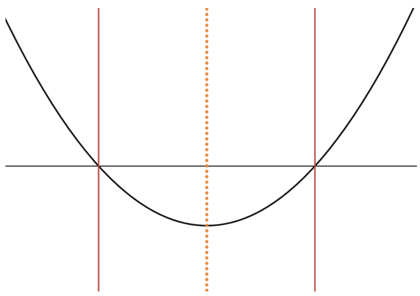

Figure 3: Quadratics have a root sequence of 010.

Degree 3

Once we move up to a cubic equation where $n = 3$ we get to have a little more freedom with where we place things because we have more roots to work with. We add another 0, another 1, and we add a 2 which is represented by the blue dashed line and is the root of the second derivative of our cubic equation. Here we have a 012010 root sequence, but we can actually have another root sequence now as well. On the right, we see that we can also have a 010210 root sequence as well with our cubic equation which is the same as the last, but with the 2 swapping sides of the 0. We know that by Rolle's theorem we must keep the 2 between the 1's, but we can move our 2 on either side of the 0 in the middle. There also must be a 1 between each two 0's, so we cannot move those around.

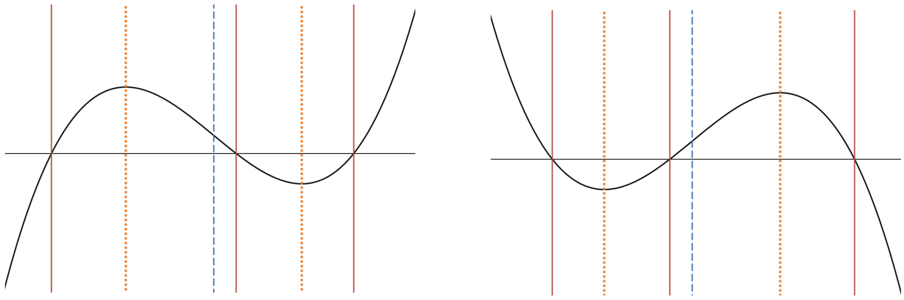

Figure 4: Cubic polynomials with root sequences 012010 and 010210

Degree 4

As we move to degree 4, there are quite a few more arrangements of roots that we can construct. By Rolle's theorem, all degree four root sequences must contain: 0101010 and 12121 and 232.

We see that there are twelve ways to do this. However, the last line of sequences as indicated in red are instances of something new, interesting, and unexpected.

0 1 2 3 0 1 2 0 1 0	0 1 0 2 1 0 3 2 1 0
0 1 2 3 0 1 0 2 1 0	0 1 2 0 1 0 3 2 1 0
0 1 0 2 1 3 0 2 1 0	0 1 2 0 3 1 2 0 1 0
0 1 0 2 1 3 2 0 1 0	0 1 0 2 3 1 2 0 1 0
0 1 2 0 1 3 0 2 1 0	0 1 2 0 3 1 0 2 1 0
0 1 0 2 3 1 0 2 1 0	0 1 2 0 1 3 2 0 1 0

Non-Realizable Root Sequences

In a theorem established by Anderson in [1], we learn that although being allowed by Rolle's theorem, the degree four root sequences 0102310210 and 0120132010 are non-realizable.

This leads to the key idea that more than Rolle's theorem determines realizable root sequences.

Root sequences that CANNOT be constructed from polynomials with all real roots are called *non-realizable*. Root sequences that CAN be constructed from polynomials with all real roots are called *realizable*.

Polynomial Root Sequence Counts

Before moving into a claim about root sequences, it will be helpful to know the specific counts of realizable and non-realizable root sequences within the respective degrees. The following table illustrates what has been previously explained by examples for degrees one to four. In [2] it is shown that there are

170 non-realizable root sequences for polynomials of degree five, however for polynomials of higher degree it is an open problem to determine the count of non-realizable root sequences. As we reach degree five, we see that the total number of root sequences allowed by Rolle's theorem quickly jumps, and by degree six it has grown exponentially. As the table shows, there are many root sequences allowed by Rolle's theorem, that are non-realizable for polynomials.

Degree	Allowed by Rolle's Theorem	Realizable	Non-realizable
1	1	1	0
2	1	1	0
3	2	2	0
4	12	10	2
5	286	116	170
6	33,592	Unknown	Unknown

Table 1: Counts of Polynomial Root Sequences

Main Result

Our main result is the following:

Theorem 1. *For any fifth degree polynomial, if its root sequence contains 113311 then it is non-realizable.*

We note that the the 0's, 2's, and 4 are suppressed in the statement of this theorem in order to allow for all possible placements of these roots that are consistent with Rolle's theorem. We will later show that there are 40 such sequences. The remainder of this section proves our theorem.

Initial Steps

Suppose by way of contradiction that any root sequence containing 113311 is realizable for a fifth degree polynomial.

The *Root Dragging Theorem* in [1] demonstrates that "dragging" the roots of a polynomial in a given direction has a corresponding effect on the location of the roots of the derivatives of the polynomial. So, for example, if we move any subset of roots of a polynomial to the right, by the root dragging theorem, the roots of the first derivative will also move to the right.

We use the root dragging theorem in order to construct a polynomial with root sequence containing 113311 and for which the roots of its first derivative are symmetric about $x = 0$. We scale the x-axis so that these roots are $\pm 1, \pm K$, with $0 < K < 1$, and scale vertically obtain the following polynomial:

$$f'(x) = 15(x-1)(x+1)(x-K)(x+K). \qquad (1)$$

Integrating $f'(x)$ once we obtain the following fifth degree polynomial, where D is the constant of integration:

$$f(x) = 3x^5 - 5(K^2+1)x^3 + 15K^2x + D.$$

Let K and D be chosen so that this polynomial has five real roots. Additionally, we find that $f'''(x) = 180x^2 - 30(K^2+1)$ has roots $x = \pm\sqrt{(1+K^2)/6}$.

Rise/Drop Condition

In order to have a realizable fifth degree polynomial we must have five real roots. As seen in Figure 5, if all five roots of $f(x)$ are real, then $f(1) < f(-1)$, which is equivalent to $0 < f(-1) - f(1)$. We evaluate and simplify:

$$\begin{aligned} &0 < f(-1) - f(1) \\ \Rightarrow &0 < -6 + 10(K^2+1) - 30K^2 \\ \Rightarrow &0 < 4 - 20K^2 \\ \Rightarrow &K^2 < \frac{1}{5}. \end{aligned} \tag{2}$$

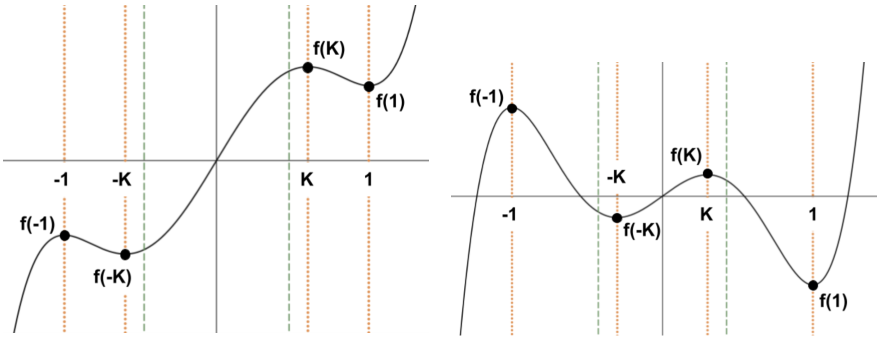

Figure 5: Left $f(-1) < f(1)$ cannot have five real roots.
Right $f(-1) > f(1)$ and has five real roots.

113311 Condition

In order for our polynomial to have a 113311 root sequence, we need to have the roots of $f'''(x)$ to be enclosed by roots of $f'(x)$. This means that

$$\sqrt{(1+K^2)/6} < K \quad \Rightarrow \quad \frac{1}{5} < K^2. \tag{3}$$

We see that Equation (2) and Equation (3) result in a contradiction. We conclude that any root sequence that contains 113311 is non-realizable for fifth degree polynomials.

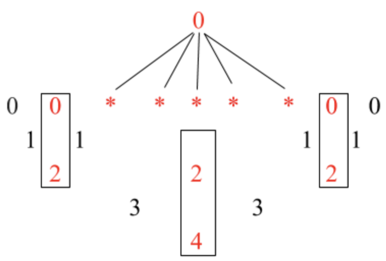

Figure 6: There are two ways to order the outer 2's and 0's, two ways to order the center 2 and 4, and there are five ways to place the middle 0.

Enumeration of the Non-Realizable 113311 Root Sequences

From the work above, we were able to show that any polynomial with the root sequence 113311 is non-realizable. We had ignored where the 0's, 2's and 4 went when doing this proof. Now that we are interested in finding the number of the 113311 non-realizable root sequences, we can look at all the possible ways to place 0's, 2's and 4's that are consistent with Rolle's theorem. If we look at Figure 6 we can see the different ways to place the 0's, 2's and 4. By a basic counting argument we find that there are a total of forty 113311 non-realizable root sequences.

We obtain 40 unique root sequences by having two ways to order the outermost 2's with adjacent 0's. There are two ways that the center 2 and 4 can be ordered. There are five places the middle 0 can go. All of these choices are independent of one another, and so we use the multiplication principle to count the different ways to order these remaining 0's, 2's and 4, giving us $2 \cdot 2 \cdot 2 \cdot 5 = 40$ total non-realizable root sequences with the 113311 configuration.

Future Work

We would like to use this proof as a model to determine the count of non-realizable root sequences of higher degree polynomials. In particular we would hope to apply this to determine the number of non-realizable root sequences for polynomials of degree six.

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Editor's Note

This work work won first place for presentations at the (virtual) 43^{rd} Biennial Convention hosted by the University of Central Missouri in Warrensburg, Missouri, April 15-17, 2021.

The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before December 31, 2022. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall 2022 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 891 - 900

Problem 891. *Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*
Let $1 \leq m,n \leq 2022$ be integers such that $(n^2 - mn - m^2)^2 = 1$. Determine the maximum value of $m^2 + n^2$.

Problem 892. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Romania.*
Find $\Omega = \int_{15\cos(2\arctan 3)}^{20\sin(2\arctan 3)} \frac{\sin^3 x + \sin^5 x}{1+\cos^2 x + \cos^4 x} dx$.

Problem 893. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Romania.*
Solve for real numbers x, y, z the equations:

$$\begin{aligned} \sin x + \sin y &= \sqrt{2 + 2\sin z} \\ \sin y + \sin z &= \sqrt{2 + 2\sin x} \\ \sin z + \sin x &= \sqrt{2 + 2\sin y}. \end{aligned}$$

Problem 894. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*
(1) Prove that for any natural number n, the polynomial

$$f_n(x) = [A(x)]^n - [B(x)]^n - [C(x)]^n + [D(x)]^n$$

is divisible by $g(x) = 2x^2 + x - 1$, where:

$$A(x) = 2x^3 + 10x^2 - 11x + 4,$$
$$B(x) = x^2 – 2x + 2,$$
$$C(x) = -x^3 + 14x^2 - 3x + 5,$$
$$D(x) = 7x3 + 8x2 + 4.$$

(2) Prove that for any natural number n, the quantity

$$\alpha_n = (-292)^n - 82^n - 1437^n + (-3068)^n$$

is divisible by 119.

(3) Prove that for any natural number n, the quantity

$$\beta_n = 65^n - 82^n - 9^n + 502^n$$

is divisible by 119.

Problem 895. *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*
Calculate the following integral:

$$\int_0^\infty \frac{\sqrt{x}\ln x}{x^4 + x^2 + 1}dx.$$

Problem 896. *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
If $a, b, c, d, x_i, y_i > 0$, for all $i = 1, \ldots, m$ and $x_{m+i} = x_i$ with $m \geq 2$, prove that

$$\sum_{k=1}^{n} \left(\sum_{i=1}^{m} \frac{(ax_i + bx_{i+1})^2}{cx_{i+2} + dy_k} \right) \geq \frac{(a+b)^2 n^2 X_m^2}{cnX_m + dmY_n}$$

where $X_m = \sum_{i=1}^m x_i$ and $Y_n = \sum_{k=1}^n y_k$.
Problem 897. *Proposed by Toyesh Prakash Sharma (student), Agra College, Agra, India.*
With F_n and L_n being the Fibonacci and Lucas numbers, show that when $n > 0$

$$\frac{{F_n}^{F_n} + {F_{n+1}}^{F_{n+1}} + {L_n}^{L_n} + {L_{n+1}}^{L_{n+1}}}{4} \geq \left(\frac{F_{n+3}}{2} \right)^{\frac{F_{n+3}}{2}}.$$

Problem 898. *Proposed by Florică Anastase, "Alexandru Odobescu" High School, Lehliu-Gară, Călăraşi, Romania.*
Let (b_n) be defined by $b_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}$ (Bătineţu's sequence). Find

$$L = \lim_{n\to\infty}\left(1+\frac{b_n}{n}\right)^{\frac{1}{n^{n-2}}\sum_{k=0}^{n}\frac{n^k}{k+1}\binom{n}{k}}.$$

Problem 899. *Proposed by Mihaly Bencze, Brasov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Solve in the set of real numbers the following equation:

$$8x^3+17x+\frac{4}{x}+\log_2^2\left(x+\frac{4}{x}\right) = x^4+20x^2+4+2^{-x^2+4x-2}.$$

Problem 900. *Proposed by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Prove that in any triangle ABC, the following inequality holds:

$$\frac{2r}{R}+\sum\frac{a^2}{bc}\geq 4.$$

SOLUTIONS TO PROBLEMS 870 - 880

Problem 870. *Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*
Find all positive integers x, y such that $x^3 - y^3 - xy = 113$.

Solution *by Kee-Wai Lau, Hong Kong, China.*

Clearly $x > y$ and we write $x = y + z$ where z is a positive integer. Then

$$113 = z^3 + y(y+z)(3z-1) \geq z^3 + 2yz(y+z) > z^3 + 2y^2.$$

Hence $z^3 < 113$ and $y^2 < \frac{113}{2}$ so that $z = 1,2,3,4, y = 1,2,3,4,5,6,7$ and $x = 2,3,4,\ldots,11$. A direct computation yields $x = 8, y = 7$ as the only solution.

Also solved by Brian Beasley, Presbyterian College, Clinton, SC; Brian Bradie, Christopher Newport University, Newport News, VA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ioannis Sfikas, National and Kapodistrian University *of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Seán Stewart, King Abdullah University of Science and Technology, Thuwal, Saudia Arabia; John Zerger, Catawba College, Salisbury, NC; and the proposer.*

Problem 871. *Proposed by Seán Stewart, Bomaderry, NSW, Australia.*
Evaluate $\int_0^{\pi/2} \csc x \log^3\left(\frac{1+\cos x+\sin x}{1+\cos x-\sin x}\right) dx$.

Solution *by Albert Stadler, Herrliberg, Switzerland.*

We perform a change of variables and put

$$y = \frac{1+\cos x+\sin x}{1+\cos x-\sin x.}$$

Then $dy = \frac{1}{1-\sin x}dx$, $\sin x = \frac{y^2-1}{y^2+1}$ and

$$\begin{aligned}
\int_0^{\pi/2} \frac{1}{\sin x}\log^3\left(\frac{1+\cos x+\sin x}{1+\cos x-\sin x}\right) dx &= \int_1^{\infty} \frac{y^2+1}{y^2-1}\log^3(y)\left(1-\frac{y^2-1}{y^2+1}\right) dy \\
&= \int_1^{\infty} \frac{2}{y^2-1}\log^3(y)dy \\
&= -\int_1^{\infty} \frac{2}{1-y^2}\log^3(y)dy \\
&= -2\sum_{k=0}^{\infty}\int_0^1 y^{2k}\log^3(y)dy \\
&= 12\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} \\
&= 12\sum_{k=0}^{\infty}\frac{1}{k^4} - 12\sum_{k=0}^{\infty}\frac{1}{(2k)^4} \\
&= 12\cdot\frac{15}{16}\cdot\frac{\pi^4}{90} = \frac{\pi^4}{8}.
\end{aligned}$$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Ankush Kumar Parcha (student) Indira Gandhi National Open University, Delhi, India; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; and the proposer.

Problem 872. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Romania.*
Let $x, y, z \in (0,1)$ with $xy+yz+zx = 1$. Prove that

$$4(x+y+z) \leq \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+9xyz.$$

Solution *by Ioan Viorel Codreanu, Satulung, Maramures, Romania.*

Let $S_1 = \frac{(\sum x)}{3}, S_2 = \sqrt{\frac{xy}{3}}$ and $S_3 = \sqrt[3]{xyz}$. Then $S_2 = \frac{1}{\sqrt{3}}$. The inequality

$$4(x+y+z) \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 9xyz$$

is equivalent to $4\left(\sum x\right)\left(\prod x\right) \leq 1 + 9(\prod x)^2$. The last inequality becomes $12S_1S_3^3 \leq 1 + 9S_3^6$. Using the Newton Inequality $S_2^2 \geq S_1S_3$, we get
$S_1S_3 \leq \frac{1}{3}$. Then $12S_1S_3^3 \leq 4S_3^2$. By the Maclaurin Inequality, we have
$S_1 \geq S_2 = \frac{1}{\sqrt{3}} \geq S_3$. Using the AM-GM Inequality, we get

$$1 + 9S_3^6 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + 9S_3^6 \geq 4\sqrt[4]{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot S_3^6} = 4\sqrt[4]{\frac{1}{3} \cdot S_3^6}.$$

It is enough to prove that $4\sqrt[4]{\frac{1}{3} \cdot S_3^6} \geq S_3^2$ which is equivalent to $\frac{1}{\sqrt{3}} \geq S_3$, and we are done.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; Marian Ursărescu, 11Roman-Vodă" National College, Roman, Romania; and the proposer.

Problem 873. *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Prove that $a^2\tan^k x + b^2\sin^k x > 2abx^k$ for all $x \in (0, \frac{\pi}{2})$ and positive integer k.

Solution *by Marian Ursărescu, "Roman-Vodă" National College, Roman, Romania*

Because $x \in (0, \frac{\pi}{2})$ we have $\tan x > 0$ and $\sin x > 0$, hence the AM-GM inequality says

$$a^2\tan^k x + b^2\sin^k x \geq 2\sqrt{a^2b^2\tan^k x\sin^k x} = 2ab\sqrt{\frac{\sin^{2k} x}{\cos^k x}}.$$

We must show

$$a^2\tan^k x + b^2\sin^k x \geq 2\sqrt{a^2b^2\tan^k x\sin^k x} = 2ab\sqrt{\frac{\sin^{2k} x}{\cos^k x}}. \tag{1}$$

But

$$\sqrt{\cos x} = \sqrt{1 \cdot \cos x} < \frac{1+\cos x}{2} \tag{2}$$

for $x \in (0, \frac{\pi}{2})$. From (1) and (2) we must show

$$\sin x > \frac{x(1+\cos x)}{2} \Leftrightarrow 2\sin x > x + x\cos x \Leftrightarrow 2\sin x - x - x\cos x > 0. \tag{3}$$

Let $f(x) = 2\sin x – x – x\cos x$. Then $f'(x) = \cos x - 1 + x\sin x$ and $f''(x) = x\cos x > 0$ for all $x \in (0, \frac{\pi}{2})$. Hence f' increases and then $f'(0) = 0$ so $f' > 0$ for all $x \in (0, \frac{\pi}{2})$. Therefore, f is strictly increasing and $f(0) = 0$ so $f > 0$ for all $x \in (0, \frac{\pi}{2})$. So (3) is true.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; and the proposers.

Problem 874. *Proposed by Abhijit Bhattacharjee, ex Msc student in BHU, India.* Prove that the equation $a^2 + (a+n)^2 = b^2$ with $a, b, n \in \mathbb{N}$ has infinitely many solutions for each n.

Solution *by Brian Beasley, Presbyterian College, Clinton, SC.*

We first show that the equation $a^2 + (a+1)^2 = b^2$ has infinitely many positive integer solutions and then use that result to prove that the given general equation also has infinitely many solutions.

To solve $a^2 + (a+1)^2 = b^2$ we rewrite the equation as $2b^2 - (2a+1)^2 = 1$ and apply the theory of Pell-type equations. Let $c = 2a+1$ and define sequences of positive integers b_k and c_k by

$b_1 = 5, b_2 = 29$ and $b_{(k+2)} = 6b_{(k+1)} - b_k$ for $k \geq 1$;

$c_1 = 7, c_2 = 41$ and $c_{(k+2)} = 6c_{(k+1)} - c_k$ for $k \geq 1$.

Then it is straightforward to verify that

$$b_k = \left(\frac{2+\sqrt{2}}{4}\right)\alpha^k + \left(\frac{2-\sqrt{2}}{4}\right)\beta^k$$

and

$$c_k = \left(\frac{1+\sqrt{2}}{2}\right)\alpha^k + \left(\frac{1-\sqrt{2}}{2}\right)\beta^k$$

where $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$. A quick computation confirms that $2b_k^2 - c_k^2 = 1$ for each positive integer k. Taking $a_k = \frac{(c_k - 1)}{2}$ for each k completes the proof.

Next, let n be an arbitrary positive integer. For each positive integer k, we define $a_{n,k} = na_k$ and $b_{n,k} = nb_k$ from above. Since $a_k^2 + (a_k+1)^2 = b_k^2$, we have

$$(na_k)^2 + (na_k + n)^2 = (nb_k)^2.$$

Hence $a_{n,k}^2 + (a_{n,k} + n)^2 = b_{n,k}^2$ for all positive integers k and n. So the original equation has infinitely many solutions in positive integers.

Also solved by Brian Bradie, Christopher Newport University, Newport News, *VA; Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; and the proposer.*

Problem 875. *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*
Calculate the following integral:

$$\int_0^\infty \frac{\arctan x}{\sqrt{2x^4+x^2+2}}dx.$$

Solution *by Henry Ricardo, Westchester Area Math Circle, New York.*

Let $I = \int_0^\infty \frac{\arctan(x)}{\sqrt{2x^4+x^2+2}}dx$ and $J = \int_0^\infty \frac{\text{arccot}(x)}{\sqrt{2x^4+x^2+2}}dx$. Then

$$\begin{aligned} I+J &= \int_0^\infty \arctan(x)\frac{\text{arccot}(x)}{\sqrt{2x^4+x^2+2}}dx \\ &= \frac{\pi}{2}\int_0^\infty \frac{1}{\sqrt{2x^4+x^2+2}}dx = \frac{\pi}{2}H. \end{aligned}$$

Noting that $\arctan(\frac{1}{t}) = \text{arccot}(t)$ for $t > 0$ and making the change of variable $x = \frac{1}{t}$ in I, we see that $I = J$, so that $I+J = (\frac{\pi}{2})H$ yields $I = (\frac{\pi}{4})H$. Now we express H in terms of a known transcendental function. We start with

$$\begin{aligned} H &= \int_0^\infty \frac{1}{\sqrt{2x^4+x^2+2}}dx \\ &= \int_0^1 \frac{1}{\sqrt{2x^4+x^2+2}}dx + \int_1^\infty \frac{1}{\sqrt{2x^4+x^2+2}}dx \\ &= \int_0^1 \frac{1}{\sqrt{2x^4+x^2+2}}dx + \int_0^1 \frac{1}{\sqrt{2t^4+t^2+2}}dt \\ &= 2\int_0^1 \frac{1}{\sqrt{2t^4+t^2+2}}dt, \end{aligned}$$

where we have let $x = \frac{1}{t}$ in the integral over $(1,\infty)$. The substitution $t = \tan\theta$ in the last integral gives us

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{2t^4+t^2+2}}dt &= \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{2\tan^4\theta+\tan^2\theta+2}}d\theta\\ &= \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{2\sec^4\theta-3\tan^2\theta}}d\theta\\ &= \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{2\sec^4\theta\left(1-\left(\frac{3}{2}\right)\sin^2\theta\cos^2\theta\right)}}d\theta\\ &= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{1}{\sqrt{1-\left(\frac{3}{2}\right)\sin^2\theta\cos^2\theta}}d\theta\\ &= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{1}{\sqrt{1-\left(\frac{3}{8}\right)\sin^2 2\theta}}d\theta\\ &= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{1}{\sqrt{1-\left(\frac{3}{8}\right)\sin^2 2\theta}}d\theta \text{ where } \theta \to \frac{\theta}{2}.\end{aligned}$$

Finally,

$$\frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{1}{\sqrt{1-\left(\frac{3}{8}\right)\sin^2 2\theta}}d\theta = \frac{1}{2\sqrt{2}}K\left(\frac{\sqrt{6}}{4}\right) = \frac{\sqrt{2}}{4}K\left(\frac{\sqrt{6}}{4}\right)$$

where K is the complete elliptic integral of the first kind. Putting these together, we find that the original integral equals $\frac{\sqrt{2}\pi}{8}K\left(\frac{\sqrt{6}}{4}\right)$. Numerically, this is about 0.9778.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Seán Stewart, King Abdullah University of Science and Technology, Thuwal, Saudia Arabia; and the proposer.

Problem 876. *Proposed by Ankush Kumar Parcha (student), Indira Gandhi National Open University, New Delhi, India and Toyesh Prakash Sharma (student), St. C.F Andrews School, Agra, India.*
If $x = \sum_{n=1}^{\infty}\left(x^{2n}+\frac{1}{x^{2n}}\right)$ and $y = \sum_{n=0}^{\infty}\frac{1+x^{2n+1}}{x^n}$, compute the value of x^y.

Note: The authors of this problem discovered unrecoverable errors in their solution and the problem after publication of the problem.

Problem 877. *Proposed by Angel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain.*
Let x be a real number. For any positive integer n, find closed forms for the following sums:

$$S_n^e=\sum_{k\,even}\binom{n+1}{k}x^k,\quad S_n^o=\sum_{k\,odd}\binom{n+1}{k}x^k$$
$$F_n^e=\sum_{k\,even}\binom{n+1}{k}F_k,\quad F_n^o=\sum_{k\,odd}\binom{n+1}{k}F_k$$
$$L_n^e=\sum_{k\,even}\binom{n+1}{k}L_k,\quad L_n^o=\sum_{k\,odd}\binom{n+1}{k}L_k$$

where F_n and L_n are respectively the n^{th} Fibonacci and Lucas numbers defined both by the recurrence relation $u_{n+2}=u_{n+1}+u_n$ with initial values $F_0=0, F_1=1, L_0=2$ and $L_1=1$.

Solution *by Albert Stadler, Herrliberg, Switzerland.*

Let $\alpha=\left(1+\sqrt{5})\right)\big/2, \beta=\left(1-\sqrt{5})\right)\big/2$. Then $\alpha+\beta=1, 1+\alpha=\alpha^2$, and $1+\beta=\beta^2$. We have

$$\begin{aligned}S_n^e(x)=\sum_{k\ even}\binom{n+1}{k}x^k &=\frac{1}{2}\sum_{k=0}^{n+1}\binom{n+1}{k}\left(1+(-1)^k\right)x^k\\ &=\frac{1}{2}(1+x)^{n+1}+\frac{1}{2}(1-x)^{n+1},\end{aligned}$$

$$\begin{aligned}S_n^o(x)=\sum_{k\ odd}\binom{n+1}{k}x^k &=\frac{1}{2}\sum_{k=0}^{n+1}\binom{n+1}{k}\left(1-(-1)^k\right)x^k\\ &=\frac{1}{2}(1+x)^{n+1}-\frac{1}{2}(1-x)^{n+1}.\end{aligned}$$

Thus

$$\begin{aligned}F_n^e &=\frac{1}{\sqrt{5}}S_n^e(\alpha)-\frac{1}{\sqrt{5}}S_n^e(\beta)\\ &=\frac{1}{2\sqrt{5}}(1+\alpha)^{n+1}+\frac{1}{2\sqrt{5}}(1-\alpha)^{n+1}-\frac{1}{2\sqrt{5}}(1+\beta)^{n+1}-\frac{1}{2\sqrt{5}}(1-\beta)^{n+1}\\ &=\frac{1}{2\sqrt{5}}\left(\alpha^2\right)^{n+1}+\frac{1}{2\sqrt{5}}(\beta)^{n+1}-\frac{1}{2\sqrt{5}}\left(\beta^2\right)^{n+1}-\frac{1}{2\sqrt{5}}(\alpha)^{n+1}\\ &=\frac{1}{2}F_{2n+2}-\frac{1}{2}F_{n+1},\end{aligned}$$

$$\begin{aligned}F_n^o &=\frac{1}{\sqrt{5}}S_n^o(\alpha)-\frac{1}{\sqrt{5}}S_n^o(\beta)\\ &=\frac{1}{2\sqrt{5}}(1+\alpha)^{n+1}-\frac{1}{2\sqrt{5}}(1-\alpha)^{n+1}-\frac{1}{2\sqrt{5}}(1+\beta)^{n+1}+\frac{1}{2\sqrt{5}}(1-\beta)^{n+1}\\ &=\frac{1}{2\sqrt{5}}\left(\alpha^2\right)^{n+1}-\frac{1}{2\sqrt{5}}(\beta)^{n+1}-\frac{1}{2\sqrt{5}}\left(\beta^2\right)^{n+1}+\frac{1}{2\sqrt{5}}(\alpha)^{n+1}\\ &=\frac{1}{2}F_{2n+2}+\frac{1}{2}F_{n+1},\end{aligned}$$

$$\begin{aligned}
L_n^e =& S_n^e(\alpha) + S_n^e(\beta) \\
=& \frac{1}{2}(1+\alpha)^{n+1} + \frac{1}{2}(1-\alpha)^{n+1} + \frac{1}{2}(1+\beta)^{n+1} + \frac{1}{2}(1-\beta)^{n+1} \\
=& \frac{1}{2}\left(\alpha^2\right)^{n+1} + \frac{1}{2}(\beta)^{n+1} + \frac{1}{2}\left(\beta^2\right)^{n+1} + \frac{1}{2}(\alpha)^{n+1} \\
=& \frac{1}{2}L_{2n+2} + \frac{1}{2}L_{n+1},
\end{aligned}$$

$$\begin{aligned}
L_n^o =& S_n^e(\alpha) + S_n^e(\beta) \\
=& \frac{1}{2}(1+\alpha)^{n+1} - \frac{1}{2}(1-\alpha)^{n+1} + \frac{1}{2}(1+\beta)^{n+1} - \frac{1}{2}(1-\beta)^{n+1} \\
=& \frac{1}{2}\left(\alpha^2\right)^{n+1} - \frac{1}{2}(\beta)^{n+1} + \frac{1}{2}\left(\beta^2\right)^{n+1} - \frac{1}{2}(\alpha)^{n+1} \\
=& \frac{1}{2}L_{2n+2} - \frac{1}{2}L_{n+1}.
\end{aligned}$$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; and the proposer.

Problem 878. *Proposed by Mihaly Bencze, Brasov, Romania.*
Let F_n and L_n be respectively the n^{th} Fibonacci and Lucas numbers as defined above. Prove the following two inequalities:

1. $\prod\limits_{k=1}^{n}\left(\frac{2F_n}{F_k}-1\right) \geq \left(\frac{2nF_n}{F_{n+2}-1}-1\right)^n$

2. $\prod\limits_{k=1}^{n}\left(\frac{2L_n}{L_k}-1\right) \geq \left(\frac{2nL_n}{L_{n+2}-1}-1\right)^n$.

Solution *by Brian Bradie, Christopher Newport University, Newport News, VA.*

Let $x_1, x_2, \ldots, x_n \in (0,1]$, and consider the function $f(x) = \ln(\frac{2}{x}-1)$. Because $f''(x) = \frac{1}{x^2} - \frac{1}{(2-x)^2} \geq 0$ for $x \in (0,1]$, it follows from Jensen's inequality that $\frac{1}{n}\sum_{k=1}^{n} f(x_k) \geq f(\frac{1}{n}\sum_{k=1}^{n} x_k)$. This is equivalent to

$$\frac{1}{n} ln\left(\prod_{k=1}^{n}\left(\frac{2}{x_k}-1\right)\right) \geq ln\left(\frac{2}{\frac{1}{n}\sum_{k=1}^{n} x_k}-1\right),$$

or

$$\prod_{k=1}^{n}\left(\frac{2}{x_k}-1\right) \geq \left(\frac{2n}{\sum_{k=1}^{n} x_k}-1\right)^n.$$

Now

1) Let $x_k = \frac{F_k}{F_n}$ for each $k = 1, 2, \ldots, n$. Then

$$\sum_{k=1}^{n} x_k = \frac{1}{F_n}\sum_{k=1}^{n} F_k = \frac{F_{n+2}-1}{F_n}$$

and

$$\prod_{k=1}^{n}\left(\frac{2F_n}{F_k}-1\right) \geq \left(\frac{2nF_n}{F_{n+2}-1}-1\right)^n.$$

2) Let $x_k = \frac{L_k}{L_n}$ for each $k = 1, 2, \ldots, n$. Then

$$\sum_{k=1}^{n} x_k = \frac{1}{L_n}\sum_{k=1}^{n} L_k = \frac{L_{n+2}-3}{L_n}$$

and

$$\prod_{k=1}^{n}\left(\frac{2L_n}{L_k}-1\right) \geq \left(\frac{2nL_n}{L_{n+2}-3}-1\right)^n > \left(\frac{2nL_n}{L_{n+2}-1}-1\right)^n.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; and the proposer.

Problem 879. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*
Let $a > b > 0$. Evaluate $\int_0^\pi \frac{\sin^n x}{(a+b\cos x)^{n+1}}dx$ for $n = 0, 1, 2, \ldots$

Solution *by Seán Stewart, King Abdullah University of Science and Technology, Thuwal, Saudia Arabia*

Denote the integral to be found by $I_n(a,b)$ where $a > b > 0$ and n is a nonnegative integer. We shall show that $I_n(a,b) = \frac{2^n}{n!\sqrt{(a^2-b^2)^{n+1}}}\Gamma^2\left(\frac{n+1}{2}\right)$. Here Γ denotes the gamma function defined by the Eulerian integral

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du, x > 0.$$

Employing a tangent half-angle substitution of $t = \tan\frac{x}{2}$, we have $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ and $dx = \frac{2}{1+t^2}dt$, while the limits of integration are mapped from $(0,\pi)$ to $(0,\infty)$. Such a substitution produces

$$\begin{aligned} I_n(a,b) &= 2^{n+1}\int_0^\infty \frac{t^n}{[(a+b)+(a-b)t^2]^{n+1}}dt \\ &= \frac{2^{n+1}}{(a-b)^{n+1}}\int_0^\infty \frac{t^n}{\left[\left(\frac{a+b}{a-b}\right)+t^2\right]^{n+1}}dt. \end{aligned}$$

Enforcing a substitution of $t \to \sqrt{\frac{a+b}{a-b}}\sqrt{t}$, after simplifying algebraically one has

$$I_n(a,b) = \frac{2^n}{\sqrt{(a^2-b^2)^{n+1}}} \int_0^\infty \frac{t^{(n-1)/2}}{(1+t)^{n+1}} dt.$$

The integral that has appeared is a beta integral of the form

$$B(x,y) = \int_0^\infty \frac{t^x}{(1+t)^{x+y}} dt$$

with $x = y = \frac{n+1}{2}$. Thus

$$\begin{aligned} I_n(a,b) &= \frac{2^n}{\sqrt{(a^2-b^2)^{n+1}}} B\left(\frac{n+1}{2}, \frac{n+1}{2}\right) \\ &= \frac{2^n}{\sqrt{(a^2-b^2)^{n+1}}} \frac{\Gamma((n+1)/2)\Gamma((n+1)/2)}{\Gamma(n+1)} \\ &= \frac{2^n}{\sqrt{(a^2-b^2)^{n+1}}} \frac{\Gamma((n+1)/2)\Gamma((n+1)/2)}{\Gamma(n+1)}, \end{aligned}$$

as announced. Here the identity relating the beta function to the gamma function $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ has been used while $\Gamma(n+1) = n!$ has also been used.

We note the expression can be further reduced based on the parity of n. If n is odd, on replacing n with $2n+1$, we have

$$I_{2n+1}(a,b) = \frac{2^{2n+1}}{(2n+1)!(a^2-b^2)^{n+1}} \Gamma^2(n+1) = \frac{2^{2n+1}(n!)^2}{(2n+1)!\,(a^2-b^2)^{n+1}}.$$

When n is even we replace n by $2n$ and get

$$I_{2n}(a,b) = \frac{2^{2n}}{(2n)\sqrt{(a^2-b^2)^{2n+1}}} \Gamma^2(n+1/2).$$

Since $\Gamma^2(n+1/2) = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!}$,

$$I_{2n}(a,b) = \frac{\pi}{2^{2n}} \binom{2n}{n} \frac{1}{\sqrt{(a^2-b^2)^{2n+1}}}.$$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; and the proposer.

Problem 880. *Proposed by Dorin Marghidanu, Colegiul National 'A.I. Cuza', Corabia, Romania.*
Let $n \geq a_k, b_k > 0$ with n and p integers ≥ 2. Prove that

$$\frac{1}{\sqrt[p]{a_1 a_2 \ldots a_n}} + \frac{1}{\sqrt[p]{b_1 b_2 \ldots b_n}} \geq \frac{2\sqrt[p]{2^n}}{\sqrt[p]{(a_1+b_1)(a_2+b_2)\ldots(a_n+b_n)}}$$

Solution *by the proposer.*

We use the AM-GM inequality twice and get

$$\frac{1}{\sqrt[p]{a_1 a_2 \ldots a_n}} + \frac{1}{\sqrt[p]{b_1 b_2 \ldots b_n}} \geq 2 * \sqrt{\frac{1}{\sqrt[p]{a_1 a_2 \ldots a_n} * \sqrt[p]{b_1 b_2 \ldots b_n}}}$$

$(1^{st}$ time)

$$= \frac{2}{\sqrt[p]{\sqrt{a_1 b_1} * \sqrt{a_2 b_2} * \ldots \sqrt{a_n b_n}}}$$

$$\geq \frac{2}{\sqrt[p]{\frac{a_1+b_1}{2} * \frac{a_2+b_2}{2} * \ldots \frac{a_n+b_n}{2}}}$$

$(2^{nd}$ time)

$$= \frac{2^p\sqrt{2^n}}{\sqrt[p]{(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)}}.$$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Toyesh Prakash Sharma (student), St. Andrews School, Agra, India; Albert Stadler, Herrliberg, Switzerland; and Daniel Văcaru, Colegiuo Economic "Maria Teiuleanu", Piteşti, Romania.

Citation for Dr. Mark Hamner

The George R. Mach Distinguished Service Award Recipient

April 17, 2021

The George R. Mach Distinguished Service Award is presented each biennium to an individual who has made major contributions to the Society. Nominations are solicited from the chapters and the National Council determines the recipient. The chapter with which the recipient is affiliated receives a monetary award of $500.

I am pleased to announce that the recipient of the George R. Mach Distinguished Service Award for this biennium is Dr. Mark Hamner.

Dr. Hamner is the Vice Provost for Institutional Research and Improvement at Texas Woman's University in Denton, Texas. He has been the corresponding secretary of the Texas Gamma chapter for the past 20 years. He rejuvenated a chapter that had only one initiation in the fifteen years prior to his arrival. The chapter also hosted a convention for the South Central region.

Dr. Hamner served on the National Council as Secretary for ten years. During this time he updated the database and initiation report form which made the certificate creation process easier. His skills with the database also made the Treasurer's job easier. And in general, Mark was always a pleasure to work with.

The KME National Council would like to honor Dr. Hamner for his faithful service. For this reason, we are excited to present the George R. Mach Distinguished Service Award to Dr. Mark Hamner.

Report of the 43^{rd} Biennial National Convention

Kappa Mu Epsilon
April 16-17, 2021
Virtual Convention
Hosted by Missouri Beta
University of Central Missouri

The Forty Third Biennial Convention of Kappa Mu Epsilon was held April 16-17, 2021 and was hosted by the Missouri Beta Chapter at the University of Central Missouri. Due to the COVID-19 pandemic, many colleges and universities were under travel restrictions, so the convention was held virtually. A company called MathDept.org helped facilitate the meetings using the Sococo online workplace platform.

Thursday, April 16, 2021

A virtual reception and mixer took place from 7:00-9:00 p.m. At 7:00 representatives of MathDept.org presented a quick tutorial to help participants navigate the Sococo platform. Participants quickly learned their way around and soon the virtual games began. Members of the Missouri Beta chapter hosted four different game "rooms."

- Escape Room
- Kahoot (math and KME quiz game)
- scribble.io (an online drawing game)
- Math Bingo

The National Council and regional directors met by Zoom earlier in the week, so as not to miss any of the fun at the Mixer.

Friday, April 17, 2021

First General Session

The first general session began at 8:30 on Friday morning. Brian Hollenbeck, National KME president, welcomed all participants. Rhonda McKee (substituting for Steve Shattuck, National KME Secretary) called the roll. At the time of roll call, 55 students and 46 faculty members were present, representing 21 chapters. Other students and faculty joined the meetings at various times throughout the convention. There were 135 registrants from 36 different chapters. Included among the registrants were six previous national officers.

After Roll Call, President Hollenbeck introduced the national officers and any past officers who had joined the virtual meeting.

Katherine Kime, chair of the Nominating Committee, introduced the candidates for president elect and historian. Scott Thuong, of Kansas Alpha chapter, was the only candidate for president elect, and Mark Hughes, Maryland Delta, was the only candidate for historian. Brief biographies of the candidates was distributed before the convention and was available online during the convention. President Hollenbeck called for nominations from the floor. Seeing none, nominations ceased.

Paper Session #1

President Elect, Don Tosh presided over the paper sessions. The first session of student presentations began at 9:00. Presentations during this session were:

- *Square Rooting Logic: The Philosophy Behind Mathematics*, by Dakota Heathcock, Alabama Theta, Jacksonville State University
- *The VICCard Cipher: Our Contribution to the Field of Playing Card Cryptography*, by Isaac Reiter, Pensylvania Epsilon, Kutztown University
- *Improve the Accuracy of Tuberculosis Detection from Chest X-ray Using Transfer Learning*, by Christopher McHeffey, Rhode Island Beta, Bryant University

Workshops

At 10:10 a.m., after a short break, participants could choose between three faculty-led workshops, which were as follows:

- *The Hyperreal Numbers* by C. Bryan Dawson, Tennessee Gamma, Union University
- *A study of teaching methodologies and their impact on varied audiences* by Shamita Dutta Gupta, New York Kappa, Pace University
- *Building an Image Recognition Model from Scratch* by Son Nguyen, Rhode Island Beta, Bryant University

Paper Session #2

Three more students presented papers during the second session.

- *Composition of Magic Squares*, by Lindsay Moyer, Pennsylvania Epsilon, Kutztown University
- *Forbidden Posets for a Class of Interval Orders*, by Tony Glackin, Iowa Gamma, Morningside College
- *Cryptonomics: An Empirical Analysis of the Internal and External Performance of Blockchain Networks to Determine Their Changes in Price*, Luis Sanchez Mercedes, Rhode Island Beta, Bryant University

After the second paper session, a “Brady Bunch-style” group photo was taken, which was followed by a lunch break. During the lunch break, participants could drop into various “rooms” in the Sococo platform to chat with participants from other chapters.

Keynote Address
At 1:30 p.m., Joe Gallian of the University of Minnesota Duluth gave the keynote address titled, *Using Mathematics to Create Symmetry Patterns*.

Paper Session #3
The following students presented papers during the third paper session.

- *Non-realizable Polynomial Root Sequences*, by Laura Batts and Megan Moran, Indiana Gamma, Anderson University
- *Integrating Factors Throughout Differential Equations*, by Joshua Gottlieb, Pennsylvania Upsilon, Seton Hill University
- *The Graphing Calculator and Achievements in Middle School Mathematics: The Effects of Course Level and Student Achievement Level*, by Stephen Bates, New York Omicron, St. Joseph's College

Breakout Sessions and Committee Meetings
At 3:50 p.m., the Awards and Resolutions Committees met. Other participants attended either the student or faculty breakout session. The student session was led by Alex McClendon, president of the host chapter. The faculty session was led by KME President Brian Hollenbeck. Both groups discussed the pros and cons of a virtual meeting and shared ideas for chapter meetings.

Concluding Business Meeting
President Hollenbeck presided over the final business meeting. He encouraged everyone to complete the convention survey, which was sent out by email to all participants.

The following national officers gave brief reports. Electronic copies of those reports were distributed by email.

John Snow, Webmaster
Doug Brown, Pentagon Editor
Cynthia Huffman, Historian
David Dempsey, Treasurer
Rhonda McKee (For Steve Shattuck), Secretary
Don Tosh, President Elect
Brian Hollenbeck, President

Alec McClendon gave a report on the student breakout session and Rhonda McKee gave a report on the faculty breakout session. Both groups agreed that they missed some aspects of an in-person meeting, like traveling with their chapters, but given the circumstances thought this virtual meeting went well. It was agreed that it would be a good idea to offer some virtual regional conventions in the future for those who can't travel.

Gaspar Porta gave a report of the Resolutions Committee, which consisted of Gaspar Porta, Kansas Delta, Chair; Mitch Keller, Iowa Gamma; Stephen Bates, New York Omicron; Tony Glackin, Iowa Gamma; and Steven Silvestri, New York

Omicron. The report is attached.

Dianne Twigger gave a report of the Auditing Committee, which consisted of Dianne Twigger, Missouri Theta, chair; Jared Burns, Pennsylvania Upsilon; Jamye Curry, Georgia Zeta; Samiha Ashraf, Georgia Zeta; Brandt Billeck, Pennsylvania Upsilon; and Hannah Tower, Missouri Theta. The report is attached.

President Hollenbeck thanked the Nominating Committee for their work. This committee consisted of Katherine Kime, Nebraska Beta, chair; John Diamantopoulos, Oklahoma Alpha; Mark Hamner, Texas Gama; and Marcus Shell, Alabama Theta.

President Hollenbeck thanked Cynthia Huffman for her years of service to KME, and indicated that a plaque would be mailed to her in honor of her service as KME National Historian.

Since there was only one candidate for each position, Mark Hughes (historian) and Scott Thuong (president elect) were elected by acclamation. President Hollenbeck installed the new officers, including Don Tosh, who moved from president elect to the president position.

New President Tosh thanked Brian Hollenbeck for his years of service to KME and indicated that a plaque would be forthcoming.

Paper Awards

Don Tosh then introduced the Paper Awards Committee, which consisted of Don Tosh, Missouri Theta, chair; Tim Flood, Kansas Alpha; Mark Hughes, Maryland Delta; Bailey Brewer, Maryland Delta; and Peter Russell, Missouri Theta. President Tosh thanked the committee for their work and then announced the awards for top papers. The awards were:

- Third place: Lindsey Moyer, Pennsylvania Epsilon, *Composition of Magic Squares*
- Second place and People's Choice Award: Isaac Reiter, Pennsylvania Epsilon, *The VICCard Cipher: Our Contribution to the Field of Playing Card Cryptography*
- First place: Laura Batts and Megan Moran, Indiana Gamma, *Non-realizable Polynomial Root Sequences*

Mach Award

President Hollenbeck presented the George R. Mach Distinguished Service Award to Mark Hamner of the Texas Gamma chapter. Dr. Hamner is the Vice Provost for Institutional Research and Improvement at Texas Woman's University in Denton, Texas. He has been the corresponding secretary of the Texas Gamma chapter for the past 20 years. He rejuvenated a chapter that had only one initiation in the fifteen years prior to his arrival. The chapter also hosted a convention for the South

Central region.

Dr. Hamner served on the National Council as Secretary for ten years. During this time he updated the database and initiation report form which made the certificate creation process easier. His skills with the database also made the Treasurer's job easier. The Mach Award was presented to Dr. Hamner in recognition of his faithful service to KME.

McKee-Tosh Award

President Hollenbeck announced the formation of a new award to recognize outstanding chapters. A successful chapter requires a committed corresponding secretary or faculty advisor as well as students who are willing to be active on their campus and at annual conventions. So each biennium we will honor a chapter who helps make KME a vibrant honor society. The chapter will receive a plaque and a $500 stipend. The inaugural award was presented jointly to the Missouri Beta and Missouri Theta chapters. Each of these chapters regularly submits chapter reports and initiation reports in a timely manner. Each has attended every regional and national convention since at least 2000, with many students presenting papers and winning awards. Each chapter has hosted or co-hosted multiple conventions at the regional and national level. They are both quite active at the local level. Finally, each chapter has been led by a corresponding secretary for 36 years, and the award is named in honor of those corresponding secretaries, Rhonda McKee and Don Tosh.

Call for 2023 Hosts

The convention ended with a call for hosts for the 2023 convention.

Rhonda McKee
For Steven Shattuck, National Secretary

Report of the National President

This is my final convention report after eight years on the council. Serving on the National Council has been a highlight of my career and I owe that to everyone that I have been able work with during this time. KME is blessed to have so many great volunteers who are willing to give their time and energy to keep KME strong moving forward.

Unfortunately, one of those volunteers will be absent from our convention this weekend as Secretary Steve Shattuck is battling cancer. We miss him and I encourage us all to keep him in our thoughts and prayers. Of course, this past year has been tough for all of us as the pandemic has affected all our lives in small and large ways. Even KME has experienced a loss of sorts as all five of our regional conventions were cancelled last spring. As far as I am aware, 1994 was the only other year five regional conventions were held.

However, there is reason for optimism as many chapters found ways to stay active. Many chapters continued to host initiations, even if done virtually. And we installed two new chapters this biennium, with another to be installed soon afterwards:

- Minnesota Alpha chapter at Metropolitan State University in St. Paul. Installed by Brian Hollenbeck via Zoom on April 20, 2020.
- Missouri Omicron chapter at Rockhurst University in Kansas City. Installed by Brian Hollenbeck via Zoom on November 13, 2020.
- Arkansas Gamma chapter at Harding University in Searcy. Scheduled to be installed by Don Tosh on April 27, 2021.

The National Council discussed several items during this 45^{th} Biennium. Here are some highlights:

- In honor of KME's 90^{th} anniversary, we submitted a proposal for an MAA session for the 2021 Joint Mathematics Meetings which was accepted. We planned to use this as an opportunity to advertise KME and gather together as a society. We withdrew the proposal when the meeting was shifted to online only.
- We investigated the pros and cons of adding international chapters.
- We invited mathematics departments to submit a petition to join KME. We offered a reduced rate in honor of our 90^{th} anniversary.
- The final box of presidential files was shipped to the Archives of American Mathematics at the University of Texas at Austin. Before shipping them, I summarized the highlights of the council minutes and other correspondence into a logbook that can be passed to future presidents of KME. My hope is that it will be a more "digestible" version of all those files. Nevertheless, we retain electronic copies of all relevant material which can be found on the Council's Google Drive.

I also attended the ACHS meeting in Feb. 2020. Here were a few takeaways:

- Honor societies should think about how we can help Gen Z students face the unique challenges that this generation will be dealing with.
- Honor societies should be aware of the upcoming enrollment "cliff" in 2026.
- Honor societies should have succession plans in place.

I also want to recognize our outstanding regional directors, who are already starting to think about the conventions in 2022. Pete Skoner is retiring and we will certainly miss his 33 years of service to KME. Fortunately, the Great Lakes region is in good hands as Adam Fletcher of West Virginia Alpha is replacing him.

I encourage everyone to check out the most recent issue of *The Pentagon* online and our social media accounts. Many thanks to Editor Doug Brown and Social Media Coordinator Vanessa Williams for making this possible. I appreciate their efforts, along with Problem Corner Editor, Pat Costello, and the referees.

I wish outgoing Historian, Cynthia Huffman, all the best. I believe she has now served a total of 12 years on the Council. I have had the pleasure of being a part of six of those. I can always count on Cynthia to have good input and quick feedback when I need it.

I also want to recognize Don Tosh's appointment as President-Elect to fill the vacancy left when Leah Childers resigned. This is a five-year commitment in addition to the many years Don has already served. And Rhonda McKee has been subbing for Steve while he recovers. Although neither could have guessed at the time, their service resulted in them being responsible for planning the first virtual convention in KME's history. All our prior experience with conferences was inadequate, but they rose to the challenge. KME is deeply indebted to each of them for stepping up when needed. I can't imagine how we would have filled either role otherwise.

Finally, it gives me great pleasure to announce the creation of a new award. A few years ago, the council discussed the idea of recognizing outstanding chapters. While the Mach Award often rewards service at the national level, we wanted to recognize the efforts of those at a local level. A successful chapter requires a committed corresponding secretary or faculty advisor as well as students who are willing to be active on their campus and at annual conventions. So each biennium we would like to honor a chapter who helps make KME a vibrant honor society. The chapter will receive a plaque and a $500 stipend. Furthermore, we decided to name the award after corresponding secretaries at two chapters that have particularly exemplified this ideal. Let me describe them for you:

Each chapter regularly submits chapter reports and initiation reports in a timely manner. Each has attended every regional and national convention since at least

2000, with many students presenting papers and winning awards. Each chapter has hosted or co-hosted multiple conventions at the regional and national level. They are both quite active at the local level. Finally, each chapter has been led by a corresponding secretary for 36 years. This is why I am pleased to award the inaugural McKee-Tosh Outstanding Chapter award to Missouri Beta and Missouri Theta.

It is an interesting coincidence that Don (and later Rhonda) joined the council after we made the decision. It is certainly a testimony to their dedication! But it meant we have been unable to divulge these plans for the past year. Therefore, the next presidency will need to work out the process of determining future winners of this award.

Brian Hollenbeck
National President

Report of the National President-Elect

I am completing less than a full year as President Elect of KME. Leah Childers was unable to complete her term and the National Council asked if I would complete her duties and then continue as President for the next four years. After careful consideration I agreed, although I am sure there are very qualified people who could perform just as well. I am assuming that with the recent trend in ecology the Council decided not to use new officers but to recycle old officers.

The President-Elect is responsible for organizing a considerable portion of the National Convention, and preparation this year amidst a global pandemic was particularly challenging. At first we tried to have a hybrid convention, but as April approached it became clear we would have to go entirely virtual. A Call For Papers and a Call For Workshops received very little response and I, and the rest of the Council, were trying our best not to panic. But KME came through for us in a big way, although you could have come through just a little bit sooner. We ended up with 9 papers being accepted for presentation and 3 workshops being offered. Altogether there were 129 registrants for the Convention, representing 38 different chapters of KME. And the virtual nature of the meeting allowed some chapters to attend a National Convention for the first time – notably Hawaii Alpha. So what could have been a disaster turned out to be an amazing success. And I give the credit to the members and faculty sponsors of this organization. KME has always been blessed with a solid membership of good people.

I also need to thank the other Council members who had to gently remind me of what my responsibilities were and how to go about them. Thank you to Brian Hollenbeck, David Dempsey, Cynthia Woodburn, John Snow, Steve Shattuck, and Rhonda McKee. Rhonda is not on the Council but she is filling in for Steve who is recovering from recent illness and surgery. Working with these people reminded me why I was willing to return to office in KME. These are indeed quality people.

Don Tosh
President-Elect

Report of the National Secretary

The National Secretary receives all initiation reports from chapters, makes a record of those reports, up-dates mailing list information for corresponding secretaries and forwards copies of the reports to other officers. At the beginning of each new biennium, the secretary prepares a new KME brochure. In the fall of each year, the secretary send out supplies to each chapter. The supplies include information brochures and membership cards. The National Secretary also takes minutes of all business meetings of the organization and writes a report of the national convention. When a college or university petitions for a new chapter of KME, the National Secretary sends out a summary of the petition, prepared by the president, to each chapter and receives the chapter ballots.

During this, the 45^{th} Biennium of Kappa Mu Epsilon, 1,464 new members were initiated in 107 chapters. Fifty-six active chapters did not report an initiation in this biennium. There are 162 active chapters of KME and 45 inactive chapters for a total of 207 chapters. The total membership in KME at the end of the 45^{th} biennium was 89,724.

Rhonda McKee for Steven Shattuck
National Secretary

Report of the National Historian

It is with sadness that I submit this report at the end of my service as the National Historian of the Kappa Mu Epsilon National Mathematics Honor Society and 12 years on the National Council. It has been a pleasure to serve with the national and regional officers, with the corresponding secretaries and faculty sponsors of individual chapters, and with students. Kappa Mu Epsilon is a great organization which has touched the lives of many people.

The primary function of the national historian is to solicit, collect, maintain and compile records of chapter activities, installation of new chapters, and other society activities of historical significance. Chapter News reports for fall and spring semesters are solicited via email from corresponding secretaries. This regular messaging provides the opportunity to learn about chapter leadership changes and to maintain communication between the national society and the local chapters, hopefully helping to fulfill the mission of the organization to recognize students who excel in mathematics. The work of the National Historian is impossible without the aid of the corresponding secretaries for each chapter. Thank you for all that you do in serving the students at your institution, your local chapter, and the national organization!

During the biennium, I continued maintaining history records in the KME Google Drive and collection chapter reports each semester. This past summer, I received six large boxes containing the hard copy Historian files. Since then, I have reorganized the files and merged them with archived Secretary and President files. Although time-consuming, this process was extremely helpful in the creation of a 90^{th} Anniversary *KME History and Information* Booklet which is being mailed to chapters this spring.

A special thank you goes to the editor of *The Pentagon*, Doug Brown of the North Carolina Zeta Chapter. The edited Chapter News section is sent to the editor after each semester, and Doug has been amazing to work with. Also special thanks to all of the Executive Council members that I have served with during my 12 year tenure on the council. KME is blessed to have such devoted leadership! Thank you all for your service, commitment, communication, and friendship. The passion and time you give contributes to this great organization that will celebrate our 90^{th} anniversary since our founding on April 18, 1931, as we gather for our 44^{th} national convention. This national convention happens to be our first virtual convention. (Just to clarify, this would have been our 45^{th} national convention in 90 years, but two were not held during WWII, and then for the first time in 2014 we held a convention in an even-numbered year.)

Who knew when I was initiated into KME as a student in 1983 that I would have the opportunity to serve this wonderful organization at the local level as a corresponding secretary and faculty sponsor for many years, at the regional level as the North Central Regional Director for 8 years, and at the national level for 12 years (8 as Treasurer and 4 as Historian)! Maybe some of the students attending this convention will one day be national KME officers, too!

Cynthia J. Huffman
National Historian

Report of the National Treasurer
43rd Biennium (March 16, 2019 – April 6, 2021)

A Biennium Asset Report and Biennium Cash Flow Report are given below. The Asset Report shows biennium assets of **$112,857.74**. The Cash Flow Report shows that we have an **asset gain of $5,897.18** this biennium.

BIENNIUM ASSET REPORT

Total Assets at beginning of 42nd **Biennium (March 16, 2019) $106,960.56**

Current Assets (Wells Fargo Bank)		
Checking	31,866.66	
Savings	40,231.19	
Time Account xxx7120	10,100.55	
Time Account xxx7138	10,169.00	
Time Account xxx7146	10,245.17	
Time Account xxx7153	10,245.17	
Total Current Assets (as of April 6, 2021)		**$112,857.74**
- (no uncleared transactions)		- $0.00
Total Current Register Assets (as of April 6, 2021)		$112,857.74
- Activity since end of 435r Biennium (April 6, 2021)		- $ 0.00
Total Assets (Register) at end of 43rd **Biennium**		**$112,857.74**

BIENNIUM CASH FLOW REPORT

Receipts		
Initiation fees received	30,805.00	
Installation fees received	640.00	
Interest income	408.01	
Gifts & misc. income	552.02	
Total Biennium Inflow		**$32,405.03**

Expenditures		
Association of College Honor Societies	2,681.11	
Administrative expenses	3,745.05	
National Convention expenses	17,323.85	
Regional Convention expenses	0.00	
Council Meetings	0.00	
Initiation expenses:Certificates, jewelry	2,086.36	
Installation expenses	662.48	
Miscellaneous	0.00	
Total Biennium Outflow		**$26,507.85**
Biennium Cash Flow		**$ 5,897.18**

The cash flow last biennium (2017–19) was **$1,779.60. Both receipts and expenditures were down** this biennium, mainly due to the COVID-19 pandemic. Major differences include: (1) Initiation fees were down $13,550 (–30%); (2) National Convention (2019) expenditures were up $3625.89 (+26%) due mainly to increased officer travel expenses (more officers attended; higher travel expenses to MD vs MO); (3) Regional Conventions were cancelled in 2020 ($0.00 spent vs $5697.05 in 2018); (4) Initiation expenses were down $11,417.95 (–84%) because fewer certificates were printed, but mainly because the jewelry (pin) supply was replenished in bulk last biennium ($8870.98). The National Council is likely to continue to pass on part of the increase in net revenue to the students and chapters by supporting travel to future National and Regional Conventions. We have easily continued to meet our National Council goal of maintaining assets of at least $40,000. **The financial condition of Kappa Mu Epsilon is sound.** We have additional assurance by maintaining a fidelity bond insuring Kappa Mu Epsilon, Inc., against any losses resulting from dishonesty of the 5 main officers (however unlikely). I want to thank my colleagues on the National Council for their untiring dedication, as well as the corresponding secretaries who maintain such a vital role in Kappa Mu Epsilon. I am grateful for the opportunity to serve with such outstanding individuals in encouraging and recognizing students for accomplishments in mathematics.

David Dempsey
KME National Treasurer

Report of *The Pentagon* Editor

Introduced in 1941, *The Pentagon*, is the official publication of Kappa Mu Epsilon. Publication of student papers continues to be the focus of *The Pentagon*. Following tradition, papers given "top" status and other recognition by the Awards Committee at the KME National Convention are guaranteed an opportunity to be published. *The Pentagon* is now completely electronic and available for free online via the KME website:

`www.kappamuepsilon.org`.

I have been the Editor of *The Pentagon* since November 2016. The typical duties involve corresponding with authors of potential articles for submission and facilitating referee feedback and author corrections for upcoming issues. Since the last national convention, four issues of *The Pentagon* (Fall 2018, Spring 2019, Fall 2019 and Spring 2020) have been published and made available on the website. The Problem Corner continues to be made available on the KME website ahead of publication of the full issue.

The Fall 2020 issue is behind schedule for two reasons:

1. The editor desperately trying to get ahead of offering courses online due to the pandemic, and
2. A dearth of articles submitted for publication.

As of March 2021, there are a sufficient number of articles approved by the referees to produce the Fall 2020 issue, which should be completed over the coming summer. There are as yet no articles available for the Spring 2021 issue. A regular publication schedule requires a steady stream of articles from which to select, so assistance from the chapters in encouraging students to submit papers will be greatly appreciated. Students presenting at this conference and any of the regional conferences are especially urged to submit their work to *The Pentagon*.

The publication of *The Pentagon* would not be possible without the dedication of the referees, whose thoughtful reviews have been invaluable in helping authors fine-tune their submissions.

Stephen Andrilli	LaSalle University
Peter Chen	The University of Mary Hardin–Baylor
Chip Curtis	Missouri Southern State Univ.
Tara Davis	Hawaii Pacific University
Vincent Ferlini	Keene State College
Adam Fletcher	Bethany College
Tim Flood	Pittsburg State Univeristy
Mark Hughes	Frostburg State University Thomas Kent Marywood University
Tom McNamara	Southwestern OK State Univ.
J. Lyn Miller	Slippery Rock University
Lloyd Moya	Henderson State University
Ann Podleski	Harris-Stowe State Univ.
Sara Quinn	Dominican University
Adam Salminen	University of Evansville
Manyiu Tse	Molloy College
John Zerger	Catawba College

Finally, I am very grateful to the Associate Editors: Pat Costello, who organizes the Problem Corner for each issue, and KME Historian Cynthia Huffman, who collects and prepares the KME News Items, as well as KME Webmaster John Snow. Their patience and attention to detail are very much appreciated.

Doug Brown
Editor, *The Pentagon*

Report of the Audit Committee
2019-2021, 43^{rd} Biennium – Kappa Mu Epsilon

Audit Committee Members:

Dianne Twigger, Missouri Theta, faculty, chair
Jared Burns, Pennsylvania Upsilon, faculty
Jamye Curry, Georgia Zeta, faculty
Samiha Ashraf, Georgia Zeta, student
Brandt Billeck, Pennsylvania Upsilon, student
Hannah Tower, Missouri Theta, student

Audit Process

1. Treasurer David Dempsey provided electronic copies of the biennium financial summary data to the committee chair to facilitate verification of asset account totals prior to the convention. The Audit Committee Chair subsequently contacted Wells Fargo Bank (Jacksonville, Alabama) by telephone. The account balances for the Kappa Mu Epsilon Platinum Business Checking account, Platinum Business Savings accounts, and the four Time accounts were verified to correspond to the associated totals found on Treasurer Dempsey's biennium reports and current records. This verification was conducted on April 12, 2021 for the balances as submitted by Treasurer Dempsey on April 8, 2021.
2. Treasurer Dempsey provided the committee with detailed documentation for receipt and payment transactions, monthly bank account statements and reconciliation documentation, expense reports, receipts, income information, as well as his own reports and summary for the full biennium. These documents were shared electronically with committee members approximately one week prior to the convention.
3. On April 8, the Audit Committee Chair and a student corresponded with President Brian Hollenbeck and interim secretary Rhonda McKee to determine their impressions of the accuracy and completeness of the recording of the financial transactions throughout the biennium.

Findings

1. The bank information provided by Treasurer Dempsey (as of April 6, 2021) was verified by the committee chair on April 12, 2021 via a phone conversation with Mr. Blake Dalrymple of Wells Fargo Bank (Jacksonville, Alabama).
2. The committee spot-checked the Secretary's report and corresponding computer generated Treasurer's report and found no inconsistencies. The chair spoke with the secretary over email on 4/8/21 and indicated complete confidence in and satisfaction with the process.
3. The committee spot-checked the Secretary's report and corresponding computer generated Treasurer's report and found no inconsistencies. The chair spoke with the secretary over email on 4/8/21 and indicated complete confidence in and satisfaction with the process.
4. The committee inspected monthly financial institution statements, quarterly interest statements, and CD interest reports, comparing them to the Treasurer's quarterly reports. Complete consistency was found between bank statements throughout the biennium.

Recommendations

1. Information forwarded by the Treasurer to the committee chair prior to the national convention provides the opportunity for verification of assets in a careful and timely manner. This process should be continued for future audit committees.
2. The organization files an electronic tax notecard annually even though no taxes are required. The committee recommends that this practice continue.
3. The internal checks built into the regular financial processing between the Treasurer, President, and Secretary provide an important safeguard to the integrity of the office of the Treasurer and help avoid the necessity of an expensive external audit. These ongoing internal audit processes should be continued and updated by the National Council as needed.
4. The committee recommends that for future audit committees, a copy of approved policies and safeguards be provided electronically to the audit committee chair with biennium financial summary data.
5. The committee recommends that the electronic process continue in the future. Access to the materials prior to the convention provided the committee with more time to review and discuss this important task as well as ask for clarifications when necessary. For future audits, we recommended that materials be provided to the committee two weeks prior to the convention.
6. For files shared electronically (with the exception of the bank information for verification by the chair), the committee recommends that significant portions of the account information be redacted from documents for security reasons.

Commendations

1. The committee commends Treasurer David Dempsey for his exemplary maintenance, management, and presentation of the financial records, and are grateful for his continued appointment as treasurer.
2. We further commend Treasurer Dempsey for his valuable input and transparency throughout the process. His detailed written guidelines were extremely helpful for the Audit Committee, and he promptly answered any questions that arose.
3. The committee commends the national President, Secretary, and Treasurer for the manner in which they communicate and cooperate to maintain the internal checks that preserve the integrity of the office of Treasurer.
4. The committee is thankful for the sample reports provided by the Treasurer to ensure consistency of the audit process.

Dianne Twigger
Chair, Audit Committee

Report of the Resolutions Committee
43rd Biennial National Convention, April 17, 2021

The Resolutions Committee consisted of:
Gaspar Porta (faculty member from Kansas Delta chapter),
Mitch Keller (faculty member from Iowa Gamma chapter),
Tony Glackin (student member from Iowa Gamma chapter),
Stephen Bates (faculty member from New York Omicron chapter), and
Steven Silvestri (student member from New York Omicron chapter).

This Committee hereby proposes the following resolutions.

"Whereas the success of any undertaking relies heavily upon the dedication and ability of its leaders, be it resolved that this Forty-Third Biennial National Convention express its gratitude

a. to Brian Hollenbeck (national president), Don Tosh (president elect), Steve Shattuck (national secretary), David Dempsey (national treasurer), Cynthia Huffman (national historian), and John Snow (national webmaster);
b. to Rhonda McKee for filling in as acting National Secretary for the Convention;
c. to Doug Brown for his service as editor of *The Pentagon*;
d. to Donna Marie Pirich, Pete Skoner, Jamye Curry, Katherine Kime, and John Diamantopoulos for their service as regional directors; and
e. to the students and faculty who served on the Auditing, Awards, Nominating, and Resolutions committees, which is so essential for the success of the meeting."

"Whereas the primary purpose of Kappa Mu Epsilon is to encourage participation in mathematics and the development of an appreciation for its beauty, be it further resolved:

a. That students Dakota Heathcock, Isaac Reiter, Christopher McHeffey, Lindsey Moyer, Tony Glackin, Luis Sanchez Mercedes, Laura Batts, Megan Moran, Joshua Gottlieb, and Stephen Bates, who prepared, submitted, and then presented their papers be given special commendation by this Forty-Third Biennial Convention for their enthusiasm and dedication.
b. That this Convention express its thanks
    i. to workshop host C. Bryan Dawson, "The Hyperreal Numbers";
    ii. to workshop host Shamita Dutta Gupta, "A Study of Teaching Methodologies and their Impact on Varied Audiences";
    iii. to workshop host Son Nguyen, "Building an Image Recognition Model from Scratch"; and
    iv. to Joe Gallian for his outstanding keynote address "Using Mathematics to Create Symmetry Patterns" on Saturday afternoon."

“Finally, whereas the University of Central Missouri of Warrensburg, MO has provided this virtual Convention with gracious hospitality, masterfully using the Sococo and Zoom platforms, be it resolved:

a. That this Forty-Third Biennial Convention express its heartfelt appreciation to the Missouri Beta chapter for the thorough arrangements they have planned and carried out so successfully, and
b. to the American Mathematical Society and to the American Statistical Association, through whose grant support the operations of this Convention were augmented;
c. That this Convention recognize and thank Kevin Cook for this stellar technical knowledge and alacritous support together with all the other members of Missouri Beta, who devoted their time and talents to ensure the success of this meeting—including organizing the activities for the reception.”

These resolutions respectfully submitted,
Gaspar Porta, Chair

Kappa Mu Epsilon News

Edited by Mark Hughes, Historian
Updated information as of March 2022

News of chapter activities and other noteworthy KME events should be sent to

Mark Hughes, KME Historian
Frostburg State University
Department of Mathematics
Frostburg, MD 21532
or to
mhughes@frostburg.edu

KAPPA MU EPSILON
Installation Report
Georgia Theta, College of Coastal Georgia
Brunswick, Georgia

The installation of the Georgia Theta Chapter of Kappa Mu Epsilon was held at 3:00 p.m. ET on Friday, October 22, 2021, on the campus of the College of Coastal Georgia in Brunswick, GA. Dr. Aaron Yeager, Assistant Professor of Mathematics at CCGA, served as master of ceremonies and conductor. Dr. David Dempsey, KME National Treasurer, served as the installing officer via a Zoom connection, as he could not be present in person. During the ceremony, students were recognized individually, received their membership certificates, and signed the new chapter roll book. After the installation, Dr. Dempsey gave a talk on "A Brief History of ~~Time~~. . . Functions."

As Dr. Yeager was already a KME member, the following student charter members were initiated during the installation: Kaelyn N. Tyler, Marianela Landi, Ben Huynh, Mallory Boyd, Alexus Thomas, Travis Simmons, Garrett S. Moseley, Darius J. Hammond, Justin D. Von Gartzen, Monique Lynn Deschenes, Dylan Thomas Morgan, and Kayla Russo.

The first officers of the Georgia Theta chapter were installed and are as follows:

President: Marianela Landi
Vice President: Monique Deschenes
Secretary: Darius Hammond
Treasurer: Kaelyn Tyler
Corresponding Secretary and Faculty Sponsor: Dr. Aaron Yeager

Georgia Theta

Missouri Theta Celebrates 50 Years

Missouri Theta held the 50th anniversary of their installation into Kappa Mu Epsilon. The ceremony was timed to coincide with Evangel University's homecoming which was held October 7-9, 2021. Missouri Theta was installed on January 12, 1971. Dr. Glenn Bernet was the charter Corresponding Secretary, and he was able to attend the anniversary. Missouri Theta was awarded the McKee/Tosh Outstanding Chapter Award at the national convention that was held virtually on April 16-17, 2021. Dr. Joseph Bohanon, a KME member who presented two papers at past KME conventions, was selected as the outstanding alumnus of Evangel's science department and gave a talk on cryptography. Joe works for the Department of Defense. We were fortunate to have most of the full-time math teachers who taught at Evangel over the last 50 years. From left to right they are: Glenn Bernet, Bill Cook, Don Tosh, Dianne Twigger, and Duane Huechteman. It was a very memorable day!

Chapter News

AL Theta – Jacksonville State University
Chapter President – Hannah Boozer; 303 Total Members
Other Fall 2021 Officers: Bronte Ray, Vice President; Hannah Davis, Secretary; Matthew Chiaravalloti, Treasurer; Dr. David Dempsey, Corresponding Secretary; and Dr. Jason Cleveland, Faculty Sponsor.
The Alabama Theta chapter had monthly meetings, complete with games and masks (for safety). We celebrated the graduation of two officers, Hannah Boozer and Matthew Chiaravalloti, in December, so we will be electing some new officers in the spring. We look forward to our annual initiation in spring, as well.

AR Gamma – Harding University
Corresponding Secretary – Ronald Smith; 24 Total Members
Due to issues with Covid protocol, we did not do anything this fall.

CT Beta – Eastern Connecticut State University
Corresponding Secretary and Faculty Sponsor – Dr. Mehdi Khorami; 534 Total Members

GA Zeta – Georgia Gwinnett College
Chapter President – Aviva Kerven; 66 Total Members; 13 New Members
Other Fall 2021 Officers: Alexa Sheets, Vice President; Hope Doherty, Secretary; William Watts, Treasurer; Dr. Jamye Curry Savage, Corresponding Secretary and Faculty Sponsor; and Dr. Livinus Uko, Faculty Sponsor.
This semester our chapter had one KME member to graduate.

IA Alpha – University of Northern Iowa
Chapter President – Lauren Dierks; 1111 Total Members; 7 New Members
Other Fall 2021 Officers: Jacob Metzen, Vice President; Maxwell Tensen, Secretary; Lydia Butters, Treasurer; and Dr. Mark D. Ecker, Corresponding Secretary and Faculty Sponsor.
Due to the COVID-19 pandemic, only one KME meeting took place in the Fall 2021 semester for Iowa Alpha. The KME officers met with the KME faculty advisor on November 8, 2022 and decided not to hold a KME banquet at a local restaurant this semester. However, seven new members were initiated into KME this semester.

IA Gamma – Morningside University
Chapter President – Anthony Franck; 440 Total Members
Other Fall 2021 Officers: Emily Stiernagle, Vice President; Beverly Russell, Secretary and Treasurer; and Mitchel T. Keller, Corresponding Secretary and Faculty Sponsor.

IL Zeta – Dominican University
Corresponding Secretary – Mihaela Blanariu; 462 Total Members
We have not inducted any new members in fall 2021.

KS Beta – Emporia State University
Chapter President – Shelby Hettenbach; 1532 Total Members; 4 New Members
Other Fall 2021 Officers: Austin Crabtree, Vice President; Mackenzee Olson, Secretary; Ian Hull, Treasurer; Tom Mahoney, Corresponding Secretary; and Brian Hollenbeck, Faculty Sponsor.
New Initiates – Brady Johnson, Katey Dembowski, Dr. Yuling Zhuang, and Deja Kuehn.

KS Delta – Washburn University
Chapter President – Clare Bindley; 822 Total Members
Other Fall 2021 Officers: Kael Ecord, Vice President; Ajar Basnet, Secretary; Katherine Cook, Treasurer; and Sarah Cook, Corresponding Secretary and Faculty Sponsor.
Members of the Kansas Delta Chapter of Kappa Mu Epsilon assisted the Washburn Mathematics and Statistics Department with our annual Math Day, an event for high school students which features both a written test and a team "Mathnificient Race". President Clare Bindley was a co-emcee for the awards ceremony. The Chapter held three meetings in fall 2021 in conjunction with Washburn's Club Mathematica.

MD Alpha – Notre Dame of Maryland University
Chapter President – Cecia Zavala Ramos; 402 Total Members
Other Fall 2021 Officers: Christina McConnell, Vice President; Erika Kaschak, Secretary; Shawne Samaco, Treasurer; and Charles Buehrle, Corresponding Secretary and Faculty Sponsor.

MD Delta – Frostburg State University
Chapter President – Ashley Armbruster; 540 Total Members
Other Fall 2021 Officers: Madison Green, Vice President; Jessica Farrell, Secretary; Jay Collins, Treasurer; Mark Hughes, Corresponding Secretary and Faculty Sponsor; and Frank Barnet, Faculty Sponsor.
The members of Maryland Delta Chapter were very happy to resume in-person meetings this fall semester. We had three meetings where we enjoyed puzzles and pizza. We also represented the Mathematics Department at our university's annual Majors Fair. In early December, some members attended the SUMS (Shenandoah Undergraduate Mathematics and Statistics) Conference virtually. Chapter President Ashley Armbruster was among the presenters.

MI Beta – Central Michigan University
Chapter President – Kelsey Knoblock; 1758 Total Members
Other Fall 2021 Officers: Emily Naegelin, Vice President; Jenna Wazny, Secre-

tary; Jeremy Proksch, Treasurer; and Dmitry Zakharov, Corresponding Secretary and Faculty Sponsor.

This semester the Michigan Beta Chapter held 6 general meetings, a book sale fundraiser, and a volunteer tutoring event. The meetings this semester included research talks from a graduate student named Chase Miller, and from a professor, Dr. Benjamin Salisbury. The chapter officers also discussed their mathematical research and interests at a meeting. In addition, the Michigan Beta Chapter hosted a math movie night, a math Jeopardy night, and a math riddle game night.

MO Theta – Evangel University

Chapter President – Peter Russell; 294 Total Members

Other Fall 2021 Officers: Hannah Tower, Vice President; and Dianne Twigger, Corresponding Secretary and Faculty Sponsor.

We held three meetings this fall. Our traditional trip to the MAKO conference was cancelled.

NC Zeta – Catawba College

Chapter President – Maria Arnold; 99 Total Members; 11 New Members

Other Fall 2021 Officers: Abigail Hartman, Vice President; Ofek Malul, Secretary; Jackson Chapin, Treasurer; and Dr. Katherine Baker, Corresponding Secretary and Faculty Sponsor.

NE Alpha – Wayne State College

Chapter President – Lacie Cruise; 1059 Total Members; 12 New Members

Other Fall 2021 Officers: Dr. Jennifer Langdon, Corresponding Secretary and Faculty Sponsor.

The Nebraska Alpha Chapter's Fall 2021 activities included initiating 12 new members, enjoying a Secret Santa holiday celebration along with several game nights. We also hosted a speaker (jointly with Math Club) and handed out candy at Halloween.

New Initiates – Kourtney Caniglia, Halie Chinn, Lilly Dahir, Brooklyn Gierhan, Hope Gubbels, Ky Kenny, John Klemmensen, Joe Matt, Calby Ruskamp, Lily Shafer, Paris TeBrink, and Kim Vidlak.

NJ Epsilon – New Jersey City University

Corresponding Secretary and Faculty Sponsor – Dr. Alemtsehai Turasie; 151 Total Members

Other Fall 2021 Officer: Dr. Debananda Chakraborty, Faculty Sponsor.

Activities were suspended due to the pandemic.

NY Nu – Hartwick College

Chapter President – Dell Potts; 349 Total Members; 10 New Members

Other Fall 2021 Officers: Shane Lamparter, Vice President; Hannah Bochniak, Secretary; James Lukasik, Treasurer; and Min Chung, Corresponding Secretary *and Faculty Sponsor.*

OK Alpha – Northeastern State University
Corresponding Secretary and Faculty Sponsor – Mark Buckles; 1872 Total Members; 9 New Members
We intend to have at least one speaker this semester and we also usually have at least one social activity (such as an ice cream social) each semester. We don't currently have any officers because our officers graduated and we are in a rebuilding year.
New Initiates – Elliot Reif, Kerri Fischer, Luan Nguyen, Cade Clinkenbeard, Jenni McClanahan, Kyla Willever, Ashley Titsworth, Chuang Shao, and Austin Beard.

PA Pi – Slippery Rock University
Chapter President – Spencer Kahley; 140 Total Members
Other Fall 2021 Officers: Boris Brimkov, Corresponding Secretary; and Amanda Goodrick, Faculty Sponsor.
We did not have any activities in fall 2021 but we will be sending out invitations in the next couple weeks to start planning the next initiation ceremony.

PA Rho – Thiel College
Chapter President – Ethan Stishan; 139 Total Members
Other Fall 2021 Officers: Camryn Sankey, Vice President; Cassie Brown, Secretary; Kara Baumgardner, Treasurer; Dr. Russell Richins, Corresponding Secretary; and Dr. Jie Wu, Faculty Sponsor.
We met as a chapter during fall 2021, but unfortunately our main activity was canceled due to a schedule conflict. We are hoping for a better outcome in the spring.

RI Beta – Bryant University
Corresponding Secretary – Prof. John Quinn; 188 Total Members
Other Fall 2021 Officers: Prof. Gao Niu, Faculty Sponsor.
Professor Gao Niu has replaced Professor Emeritus Alan Olinsky as faculty sponsor. Alan had served as faculty sponsor since the inception of the Rhode Island Beta Chapter in 2013. He retired from Bryant in the spring of 2021. We will be soliciting and evaluating nominations for our KME chapter during the spring of 2022.

TN Gamma – Union University
Chapter President – Braden Watkins; 517 Total Members
Other Fall 2021 Officers: Joya Schrock, Vice President; Rylee Iorio, Secretary and Treasurer; Taylor Overcast, Webmaster and Historian; Bryan Dawson, Corresponding Secretary; and Matt Lunsford, Faculty Sponsor.

TX Lambda – Trinity University
Corresponding Secretary – Dr. Hoa Nguyen; 307 Total Members

WV Alpha – Bethany College

Chapter President – Amanda M. Reynolds; 194 Total Members

Other Fall 2021 Officers: Jacob C. Thornburg, Vice President; Cullen J. Wise, Secretary and Treasurer; and Dr. Adam C. Fletcher, Corresponding Secretary and Faculty Sponsor.

West Virginia Alpha, like many other chapters across the country, continued to adjust to life in a pandemic. The College remained predominately on-campus throughout the fall semester, with strict COVID protocols in place throughout. The local and national restrictions canceled or postponed a number of the chapter's usual activities, like mathematics competitions and professional gatherings. West Virginia Alpha chapter and our local Mathematics and Computer Science Club continued to attend meetings virtually and host small chess and gaming tournaments on campus. The chapter eagerly awaits the next in-person meetings as soon as they are possible.

Active Chapters of Kappa Mu Epsilon

Listed by date of installation

Chapter	Location	Installation Date
OK Alpha	Northeastern State University, Tahlequah	18 Apr 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
AL Alpha	Athens State University, Athens	5 Mar 1935
NM Alpha	University of New Mexico, Albuquerque	28 Mar 1935
IL Beta	Eastern Illinois University, Charleston	11 Apr 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 Apr 1937
OH Alpha	Bowling Green State University, Bowling Green	24 Apr 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	University of Central Missouri, Warrensburg	10 Jun 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 Jun 1941
MI Beta	Central Michigan University, Mount Pleasant	25 Apr 1942
NJ Beta	Montclair State University, Upper Montclair	21 Apr 1944
IL Delta	University of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 Mar 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 Jun 1947
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 Apr 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	University of Nebraska—Kearney, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960
OH Epsilon	Marietta College, Marietta	29 Oct 1960
MO Zeta	University of Missouri—Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood University, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 Apr 1965
AL Epsilon	Huntingdon College, Montgomery	15 Apr 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
TN Gamma	Union University, Jackson	24 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	McDaniel College, Westminster	30 May 1965
IL Zeta	Dominican University, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Truman State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel University, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 Mar 1971
KY Alpha	Eastern Kentucky University, Richmond	27 Mar 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 Apr 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973

NY Kappa	Pace University, New York	24 Apr 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State University, Joplin	8 May 1975
GA Alpha	State University of West Georgia, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin—Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sep 1978
IL Theta	Benedictine University, Lisle	18 May 1979
PA Mu	St. Francis University, Loretto	14 Sep 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C.W. Post Campus of Long Island University, Brookville	2 May 1983
MO Kappa	Drury University, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 Mar 1985
NE Delta	Nebraska Wesleyan University, Lincoln	18 Apr 1986
TX Iota	McMurry University, Abilene	25 Apr 1987
PA Nu	Ursinus College, Collegeville	28 Apr 1987
VA Gamma	Liberty University, Lynchburg	30 Apr 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 Apr 1990
CO Delta	Mesa State College, Grand Junction	27 Apr 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 Apr 1991
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 Mar 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993
MS Epsilon	Delta State University, Cleveland	19 Nov 1994
PA Omicron	University of Pittsburgh at Johnstown, Johnstown	10 Apr 1997
MI Delta	Hillsdale College, Hillsdale	30 Apr 1997
MI Epsilon	Kettering University, Flint	28 Mar 1998
MO Mu	Harris-Stowe College, St. Louis	25 Apr 1998
GA Beta	Georgia College and State University, Milledgeville	25 Apr 1998
AL Eta	University of West Alabama, Livingston	4 May 1998
PA Pi	Slippery Rock University, Slippery Rock	19 Apr 1999
TX Lambda	Trinity University, San Antonio	22 Nov 1999
GA Gamma	Piedmont College, Demorest	7 Apr 2000
LA Delta	University of Louisiana, Monroe	11 Feb 2001
GA Delta	Berry College, Mount Berry	21 Apr 2001
TX Mu	Schreiner University, Kerrville	28 Apr 2001
CA Epsilon	California Baptist University, Riverside	21 Apr 2003
PA Rho	Thiel College, Greenville	13 Feb 2004
VA Delta	Marymount University, Arlington	26 Mar 2004
NY Omicron	St. Joseph's College, Patchogue	1 May 2004
IL Iota	Lewis University, Romeoville	26 Feb 2005
WV Beta	Wheeling Jesuit University, Wheeling	11 Mar 2005
SC Epsilon	Francis Marion University, Florence	18 Mar 2005
PA Sigma	Lycoming College, Williamsport	1 Apr 2005
MO Nu	Columbia College, Columbia	29 Apr 2005
MD Epsilon	Stevenson University, Stevenson	3 Dec 2005
NJ Delta	Centenary College, Hackettstown	1 Dec 2006
NY Pi	Mount Saint Mary College, Newburgh	20 Mar 2007
OK Epsilon	Oklahoma Christian University, Oklahoma City	20 Apr 2007
HA Alpha	Hawaii Pacific University, Waipahu	22 Oct 2007
NC Epsilon	North Carolina Wesleyan College, Rocky Mount	24 Mar 2008
NY Rho	Molloy College, Rockville Center	21 Apr 2009
NC Zeta	Catawba College, Salisbury	17 Sep 2009
RI Alpha	Roger Williams University, Bristol	13 Nov 2009
NJ Epsilon	New Jersey City University, Jersey City	22 Feb 2010
NC Eta	Johnson C. Smith University, Charlotte	18 Mar 2010
AL Theta	Jacksonville State University, Jacksonville	29 Mar 2010
GA Epsilon	Wesleyan College, Macon	30 Mar 2010
FL Gamma	Southeastern University, Lakeland	31 Mar 2010
MA Beta	Stonehill College, Easton	8 Apr 2011
AR Beta	Henderson State University, Arkadelphia	10 Oct 2011
PA Tau	DeSales University, Center Valley	29 Apr 2012
TN Zeta	Lee University, Cleveland	5 Nov 2012
RI Beta	Bryant University, Smithfield	3 Apr 2013
SD Beta	Black Hills State University, Spearfish	20 Sept 2013
FL Delta	Embry-Riddle Aeronautical University, Daytona Beach	22 Apr 2014
IA Epsilon	Central College, Pella	30 Apr 2014
CA Eta	Fresno Pacific University, Fresno	24 Mar 2015
OH Theta	Capital University, Bexley	24 Apr 2015
GA Zeta	Georgia Gwinnett College, Lawrenceville	28 Apr 2015
MO Xi	William Woods University, Fulton	17 Feb 2016
IL Kappa	Aurora University, Aurora	3 May 2016
GA Eta	Atlanta Metropolitan University, Atlanta	1 Jan 2017
CT Gamma	Central Connecticut University, New Britan	24 Mar 2017
KS Eta	Sterling College, Sterling	30 Nov 2017
NY Sigma	College of Mount Saint Vincent, The Bronx	4 Apr 2018
PA Upsilon	Seton Hill University, Greensburg	5 May 2018

KY Gamma	Bellarmine University, Louisville	23 Apr 2019
MO Omicron	Rockhurst University, Kansas City	13 Nov 2020
AK Gamma	Harding University, Searcy	27 Apr 2021
GA Theta	College of Coastal Georgia, Brunswick	22 Oct 2021