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Eulerian Numbers and Relations to Number Triangles and Infinite Series

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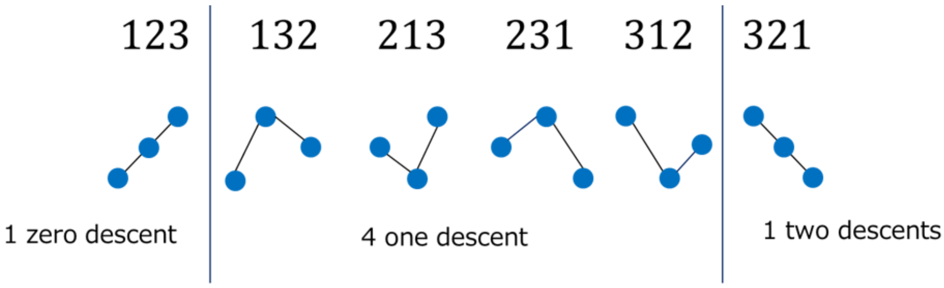
Abstract

Infinite series of the form $\sum_{i=0}^{\infty} i^n x^i$, where $n \in \mathbb{N}$, have a closed form solution that involves the Eulerian numbers, which are the coefficients of the Eulerian polynomials that were studied by Leonhard Euler. This paper will show the proof of this theorem and also a construction of two different number triangles that use the Eulerian numbers and have similarities to Pascal's Triangle.

Introduction

In combinatorics, The *Eulerian numbers* are defined as the number of permutations of the numbers 1 to n with k *descents*, or the number of elements that are smaller than the previous element. The notation $A(n,k)$ will be used to denote the Eulerian number for the permutations of 1 to n with k descents. Also, for values of $A(n,k)$ where $k \geq n > 0$ the Eulerian number is defined to be 0. These numbers are easy to compute by hand for small n, and to demonstrate we will look at the permutations from 1 to 3.

Example 1. Permutations for 1 to 3.



There are some important characteristics to notice from this example. First, the total number of permutations from 1 to n will always be equal to $n!$, so in this example $n = 3$ and 1+4+1=6=3!. Also, when there are no descents, or when $k = 0$, there is only one permutation and this is the same for the two descents case, or when $k = n-1$. This will always be the case no matter what our choice for n is, since the permutations of 1 to n will always have one that is always increasing in value and one that is always decreasing in value. For the descents between 0 and $n-1$, we will need to use a recursive formula to find all the values when n gets large.

Theorem 1. *For $1 \leq k \leq n-1$, the recursive formula for the Eulerian numbers is*

$$A(n,k) = (n-k)A(n-1,k-1) + (k+1)A(n-1,k).$$

Proof. Let $W = (w_1 w_2 w_3 \dots w_n)$ be an arbitrary permutation from 1 to n and set the descents to be $1/lek \leq n-1$. If we remove the largest element, n, from the permutation then the descents of W either stay the same or decrease by 1.

Case 1: Descents stay the same. If n is the last element of the permutation or if $W = (w_1 w_2 \dots xny \dots w_n)$, where $x > y$. In this case there are k places to remove n without changing the descent count plus the case of the last element, so we have $(k+1)A(n-1,k)$ options.

Case 2: Descents decrease by 1. If n is the first element of the permutation or if $W = (w_1 w_2 \dots xny \dots w_n)$, where $x < y$. In this case there are $n-k-1$ places to remove n that would decrease the descents by 1 plus the case of the first element, so we have $(n-k)A(n-1,k-1)$ options.

Combining these two cases we have $(n-k)A(n-1,k-1) + (k+1)A(n-1,k)$, which is the formula that was to be shown. ■

The Pattern to the Infinite Series

We will now highlight Infinite Series of the form $\sum_{i=0}^{\infty} i^n x^i$, where $n \in \mathbb{N}$, and show the patterns that arise from solving the closed form solutions for specific n. The first case will be when $n = 0$, and now the series is $\sum_{i=0}^{\infty} x^i$. This is a geometric series with the first term equal to 1, which has the well-known formula, $\frac{1}{1-x}$ for $|x| < 1$. For the $n = 1$ case, this requires manipulation of the terms of the series $\sum_{i=0}^{\infty} ix^i$. It is possible to rearrange the order of this infinite series, since we are only concerned with the values of x that make the series converge. Here is what the manipulation looks like:

Example 2. . Derivation of closed form solution of $\sum_{i=0}^{\infty} ix^i$.

$$\begin{aligned}
S_1 = \sum_{i=0}^{\infty} ix^i &= 0 + 1x^1 + 2x^2 + 3x^3 + \dots \\
&= x + x^2 + x^3 + x^4 + \dots + 1 - 1 + x^2 + 2x^3 + 3x^4 + \dots \\
&= S - 1 + x\left(1x^1 + 2x^2 + 3x^3 + 4x^4 + \dots\right),
\end{aligned}$$

where S is the geometric series $x + x^2 + x^3 + x^4 + \dots + 1 = \frac{1}{1-x}$.
Then

$$
\begin{aligned}
S_1 &= S - 1 + xS_1 \\
&\Rightarrow S_1 - xS_1 = \frac{1}{1-x} - 1 \\
&\Rightarrow S_1 - xS_1 = \frac{x}{1-x} \\
&\Rightarrow S_1(1-x) = \frac{x}{1-x} \\
&\Rightarrow S_1 = \frac{x}{(1-x)^2}.
\end{aligned}
$$

A similar but more tedious process can be repeated to find the closed forms for the series when $n = 2, 3, 4, \ldots$ and so on. Here are the closed forms for $n = 1, 2, 3,$ and 4:

$$
\begin{aligned}
S_1 &= \sum_{i=0}^{\infty} ix^i = \frac{x}{(1-x)^2} \\
S_2 &= \sum_{i=0}^{\infty} i^2x^i = \frac{x^2+x}{(1-x)^3} \\
S_3 &= \sum_{i=0}^{\infty} i^3x^i = \frac{x^3+4x^2+x}{(1-x)^4} \\
S_4 &= \sum_{i=0}^{\infty} i^4x^i = \frac{x^4+11x^3+11x^2+x}{(1-x)^5}.
\end{aligned}
$$

The important parts to look at from these solutions are the patterns that arise. First, each has a power of $(1-x)$ in the denominator that is equal to the power of the i term plus one. Also, the numerator involves a polynomial where the power of the x's ranges from the power of the i term to one. The key observation is that the coefficients of these polynomials are the Eulerian numbers. In fact, the expressions in the numerator of each solution are called Eulerian polynomials. These polynomials were studied by Leonhard Euler in his 1755 book *Institutiones Calculi Differentialis* (485-486). The definition for Eulerian polynomials that we will use, as defined by Richard P. Stanley in Enumerative Combinatorics is,"$A_d(x) = \sum_{w \varepsilon \mathfrak{S}_d} x^{1+des(w)} = \sum_{k=1}^{d} A(d,k)x^k$, where $A(d,k)$ are the Eulerian numbers for permutations with exactly $k-1$ descents, $\mathfrak{S}d$ is the group of permutations from 1 to d, and $des(w)$ is the number of descents of the arbitrary permutation w." (39). Stanley also provides a recurrence relation for these polynomials that he proves on page 40, and the recurrence is "$A_0(t) = 1, \;\; A_n(t) = t(1-t)A'_{n-1}(t) + A_{n-1}(t)(nt), \; n \geq 1$" (40).

Proof of the Closed Form Solution to the Infinite Series

We now have all the tools necessary to prove the conjecture of this paper. This proof is adapted from *Enumerative Combinatorics* by Stanley that uses induction and the recursive formula for Eulerian polynomials.

Theorem 2. $\forall n \in \mathbb{N}, \sum_{i=0}^{\infty} i^n x^i = \frac{A_n(x)}{(1-x)^{n+1}}$, *where* $A_n(x) = \sum_{k=1}^{n} A(n,k)\, x^k$ *for* $n \geq 1$.

Proof. By induction. When $n = 0, \sum_{i=0}^{\infty} x^i = \frac{A_0(x)}{(1-x)^{0+1}} = \frac{1}{1-x}$, so the base case is true.
Now suppose it is true that $\sum_{i=0}^{\infty} i^k x^i = \frac{A_k(x)}{(1-x)^{k+1}}$, for some $k \in \mathbb{N}$. Then, if we take the derivative and multiply by x on both sides we get:

$$\frac{d}{dx}(\sum_{i=0}^{\infty} i^k x^i)x = \left(\frac{d}{dx}\frac{A_k(x)}{(1-x)^{k+1}}\right)x$$

$$\Rightarrow \sum_{i=0}^{\infty} i^{k+1} x^i = \frac{(A'_k(x)(1-x)^{k+1} + A_k(x)(k+1)(1-x)^k)x}{(1-x)^{2k+2}}$$

$$= \frac{x(1-x)A'_k(x) + A_k(x)(k+1)x}{(1-x)^{k+2}} = \frac{A_{k+1}(x)}{(1-x)^{k+2}}.$$

The statement is now shown to be true for the $k+1$ case and the original statement follows from induction. ■

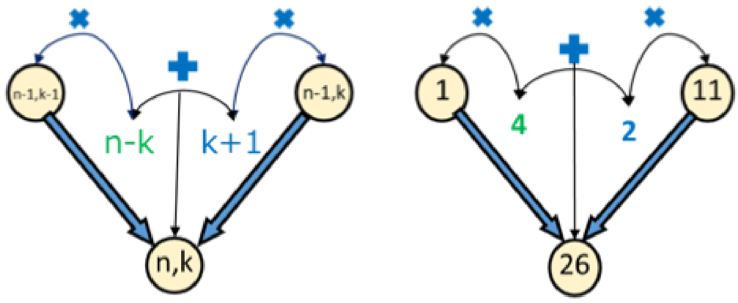
Number Triangle Constructions

The Eulerian numbers can be arranged in a triangular array that is similar to Pascal's triangle, which uses binomial coefficients. This triangular array is called Euler's triangle, and the recursive formula for generating Eulerian numbers will be useful for generating each entry. Each row of the triangle will correspond to a specific permutation of the numbers 1 to n starting with $n = 1$, and each entry in the row will count the number of descents, k, ranging from 0 to $n-1$. We will now show a picture of the triangle for the first seven rows, and a picture that will help visualize how to use the recursive formula to generate each entry.

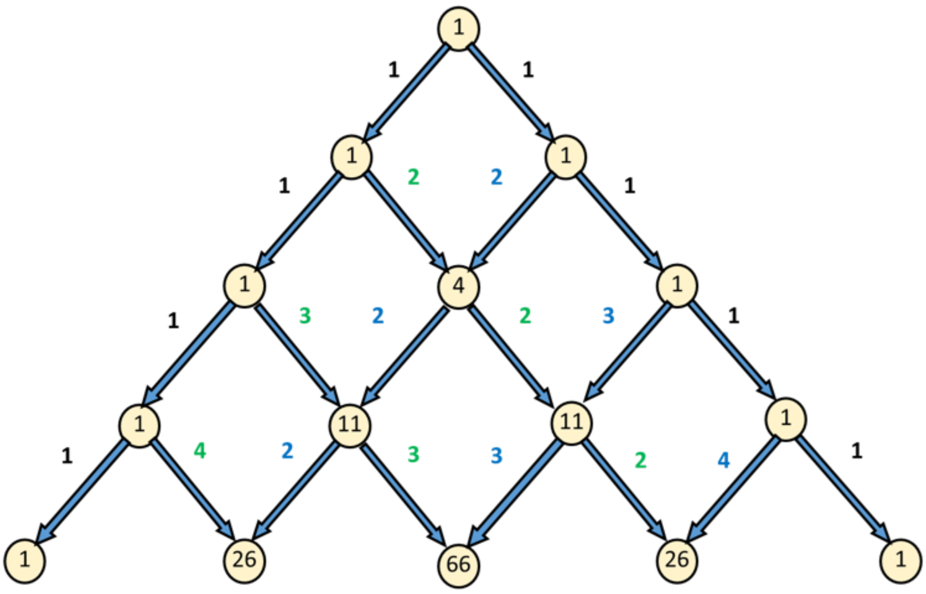
Example 3. First 7 rows of Euler's triangle:

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1
1 57 302 302 57 1
1 120 1191 2416 1191 120 1

Example 4. Visualization of how to generate each entry in Euler's triangle.



Legend



In this example we have row $n = 5$ and entry $k = 1$. The left arrow is 5-1=4 and the right arrow is 1+1=2. The $n-1, k-1$ entry is 1 and the $n-1, k$ entry is 11, so we take (1x4)+(2x11)=4+22=26. One similarity Euler's triangle has with Pascal's triangle is that the sum of each row equals a factorial number, while in Pascal's triangle each row equals a power of two.

The second number triangle we will look at is called Gillin's triangle. The first four rows of this triangle are identical to Euler's triangle but are generated differently. It also shares the property with Euler's triangle that each row adds to $n!$ beginning with row $n = 1$. The first and last entry in each row is always 1, and to get the other entries we add the $n-1, k-1$ entry and the $n-1, k$ entry plus the factorial of the $n-1$ row. This formula can be written recursively as, $G(n,k) = G(n-1,k-1) + G(n-1,k) + (n-1)!$, for $0 < k < n-1$.

Example 5. First 7 rows of Gillin's triangle:

1 = 1! = 1
1 1 = 2! = 2
1 4 1 = 3! = 6
1 11 11 1 = 4! = 24
1 36 46 36 1 = 5! = 120
1 157 202 202 157 1 = 6! = 720
1 878 1079 1124 1079 878 1 = 7! = 5040

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Alternative Proof of the Weighted Mean Value Theorem of Integral Calculus

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Abstract

In this article we present an easy proof of the Weighted Mean Value Theorem of integral calculus using Cauchy's Mean Value Theorem of differential calculus.

The Proof

In integral calculus there we have two mean value theorems. Here we provide an easy and pleasant proof of a related result, the Weighted Mean Value Theorem of integral calculus.

Theorem. *If* $f, g : [a,b] \to \mathbb{R}$ *are continuous functions with* $g(x) > 0$ *on* (a,b), *then* $\exists c \in (a,b)$ *such that* $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$.

Proof. Define $F.G : [a,b] \to \mathbb{R}$ by $F(x) = \int_0^x f(x)g(x)dx$ and $G(x) = \int_0^x g(x)dx$. Note that F and G are continuous on $[a,b]$ and differentiable on (a,b) and that $G'(x) \neq 0\ \forall x \in (a,b)$. By invoking Cauchy's Mean Value Theorem [1] we have:

$$\exists c \in (a,b) \text{ such that } \left.\frac{d}{dx}\right|_{x=c} F(x) = \frac{F(b)-F(a)}{G(b)-G(a)} \left.\frac{d}{dx}\right|_{x=c} G(x)$$
$$\Rightarrow \left.\frac{d}{d}\right|_{x=c} \int_0^x f(x)g(x)dx = \frac{F(b)-F(a)}{G(b)-G(a)} \left.\frac{d}{dx}\right|_{x=c} \int_0^x g(x)dx$$
$$\Rightarrow f(c)g(c) = \frac{F(b)-F(a)}{G(b)-G(a)} g(c)$$
$$\Rightarrow f(c) = \frac{F(b)-F(a)}{G(b)-G(a)}.$$

Here $G(b)-G(a) \neq 0$ since $g(x) > 0\ \forall x \in (a,b)$. Then, using the definitions of $F(x)$ and $G(x)$, we have:

$$f(c)\left(\int_0^b g(x)dx - \int_0^a g(x)dx\right) = \int_0^b f(x)g(x)dx - \int_0^a f(x)g(x)dx.$$

As $a < b$ then

$$\int_0^b g(x)dx = \int_0^a g(x)dx + \int_a^b g(x)dx$$

and
$$\int_0^b f(x)g(x)dx = \int_0^a f(x)g(x)dx + \int_a^b f(x)g(x)dx.$$
Then

$$\begin{aligned} f(c)&\left(\int_0^a g(x)dx + \int_a^b g(x)dx - \int_0^a g(x)dx\right) \\ &= \int_0^a f(x)g(x)dx + \int_a^b f(x)g(x)dx - \int_0^a f(x)g(x)dx \\ &\Rightarrow f(c)\int_a^b g(x)dx = \int_a^b f(x)g(x)dx. \end{aligned}$$

■

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Hausdorff Sequential Boundary

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Abstract

The sequential ends of a metric space were defined in *Sequential ends of metric spaces* by M. DeLyser, B. LaBuz, and M. Tobash and proved to be a coarse invariant. We can see that the real line and the ray are not coarsely equivalent since they have a different number of sequential ends, but the analog in two dimensions is not as easily handled. Both the Euclidean plane and half-plane have a single sequential end so this invariant does not distinguish between these two spaces.

Here we define a new sequential boundary that makes the plane (and half-plane) have infinitely many boundary points. We hope this invariant can be used to distinguish between the plane and the half-plane. In that direction we introduce a structure on the boundary. We define a coarse-invariant partial order and show that the boundary of the plane has a greatest element and infinitely many minimal elements.

Introduction

Coarse geometry is the study of metric spaces from a "zoomed out" point of view, so two spaces that look the same at a "great distance" are actually equivalent.

That is the intuitive idea of coarse equivalence. The technical definition is as follows. Given two metric spaces X and Y, we say the function $f : X \to Y$ is coarse if it is both bornologous and proper. A function is bornologous if for every $N > 0$ there is an $M > 0$ such that if $d(x,y) \leq N$, then $d(f(x), f(y)) \leq M$. The function is proper if inverse images of bounded sets are bounded. We would then say X and Y are coarsely equivalent if there are coarse functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are close to id_X and id_Y respectively. In other words, the sets $\{d(x, g \circ f(x))\}$ and $\{d(y, f \circ g(y))\}$ are bounded. If a function is a coarse function we will sometimes refer to it as a map.

To understand and visualize this idea of coarse equivalence, consider the metric spaces $\mathbb{Z}$ and $\mathbb{R}$ (Figure 1). Define $f : \mathbb{Z} \to \mathbb{R}$ to be the map $n \mapsto n$ and $g : \mathbb{R} \to \mathbb{Z}$ to be the map $x \mapsto \lfloor x \rfloor$. We are able to find that f and g are coarse functions and $f \circ g$ and $g \circ f$ are close to their respective identities, thus $\mathbb{Z}$ and $\mathbb{R}$ are coarsely equivalent.

In [2] it was noted that when a sequence (s_i) in a metric space X is viewed as a function $s : \mathbb{N} \to X$, the function being bornologous is equivalent to it being

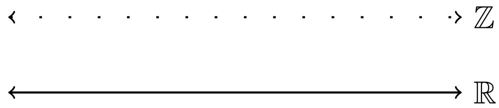

Figure 1: The integers are coarsely equivalent to the real numbers.

a K-sequence for some $K > 0$. A sequence (s_i) is a K-sequence if $\mathrm{d}(s_i, s_{i+1}) \leq K$ for all $i \in \mathbb{N}$. It is also mentioned that the function being proper is equivalent to it going to infinity. A sequence (s_i) goes to infinity if $\mathrm{d}(s_1, s_i) \to \infty$. We denote that a sequence (s_i) goes to infinity by $s_i \to \infty$.

That paper took a new view of ideas from earlier works [3], [4], and [5] where a coarse invariant was developed. That invariant was able to distinguish between the real line and a ray since the real line has two “ways of going to infinity” while the ray has just one.

The two-dimensional analog to the line and the ray is the Euclidean plane and the half plane. We conjecture that the Euclidean plane and half plane are not coarsely equivalent, but when one checks the sequential boundary of these spaces all that is found is a single point—only one way of going to infinity. In order to distinguish between these two spaces we need an invariant that detects some more detail about them.

Hausdorff Sequential Boundary

We consider (s_i) to be a coarse sequence in X if there is a coarse function $s : \mathbb{N} \to X$ such that $s(i) = s_i$ for all $i \in \mathbb{N}$. The sets $\bar{B}(x,R)$ are the closed balls $\{y \in X : \mathrm{d}(x,y) \leq R\}$.

Definition 1. *Given two coarse sequences (s_i) and (t_i) in a metric space X, we say (s_i) is Hausdorff equivalent to (t_i) if there exists $R > 0$ such that $\{s_i\} \subset \bigcup \bar{B}(t_i,R)$ and $\{t_i\} \subset \bigcup \bar{B}(s_i,R)$.*

Figure 2 shows two sequences that are Hausdorff equivalent. We call it Hausdorff equivalent because the definition is of the same flavor as that of the Hausdorff distance between metric spaces [1, Definition 5.30].

We define the Hausdorff sequential boundary of X to be the set of Hausdorff equivalence classes of coarse sequences in X, denoted as $\partial_H(X)$. We denote the Hausdorff equivalence class of a coarse sequence (s_i) by $[(s_i)]$.

To show Hausdorff equivalence is an equivalence relation, we need to show it is reflexive, symmetric, and transitive. It is obvious from the construction of the definition that it is reflexive and symmetric. To show Hausdorff equivalence is transitive, let (s_i) be Hausdorff equivalent to (t_i) and let (t_i) be Hausdorff equivalent to (q_i). Then there exist $R_1, R_2 > 0$ such that $\{s_i\} \subset \bigcup \bar{B}(t_i,R_1)$ and $\{t_i\} \subset \bigcup \bar{B}(s_i,R_1)$, and $\{t_i\} \subset \bigcup \bar{B}(q_i,R_2)$ and $\{q_i\} \subset \bigcup \bar{B}(t_i,R_2)$. We want to show (s_i) is Hausdorff equivalent to (q_i). Let $q_i \in \{q_i\}$. We know $q_i \in \bar{B}(t_j,R_2)$ for some $j \in \mathbb{N}$. We also know $t_j \in \bar{B}(s_k,R_1)$ for some $k \in \mathbb{N}$. Then $q_i \in \bar{B}(s_k,R_1+R_2)$ since

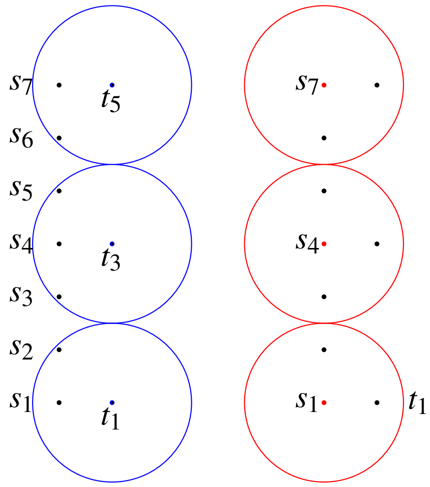

Figure 2: $\{s_i\} \subset \bigcup \bar{B}(t_i, n)$ and $\{t_i\} \subset \bigcup \bar{B}(s_i, n)$

$\mathrm{d}(q_i, s_k) \leq \mathrm{d}(q_i, t_j) + \mathrm{d}(t_j, s_k)$. Then $\{q_i\} \subset \bigcup \bar{B}(s_i, R_1 + R_2)$. A symmetric argument gives us the other inclusion, $\{s_i\} \subset \bigcup \bar{B}(q_i, R_1 + R_2)$.

We begin by showing that the Hausdorff boundary of the real line contains two elements. In other words, there are "two ways of going to infinity." In order to get that result we give a condition for a coarse sequence in $\mathbb{R}$ to be Hausdorff equivalent to the sequence of natural numbers.

Given a coarse sequence (s_i) in a metric space X, without loss of generality we can choose any point as the initial point of the sequence since given any $x \in X$, the sequences $s_1, s_2, s_3, \ldots$ and $x, s_2, s_3, \ldots$ are Hausdorff equivalent.

Proposition 1. *Let (s_i) be a coarse sequence in $\mathbb{R}$. Then $[(s_i)] = [(n)]$ if and only if (s_i) is eventually positive.*

Proof. We assume that (s_i) and (n) start at 0.

($\Rightarrow$) Suppose $[(s_i)] = [(n)]$. Then there exists $R > 0$ such that $\{s_i\} \subset \bigcup \bar{B}(n, R)$ and $\{n\} \subset \bigcup \bar{B}(s_i, R)$. Since $s_i \to \infty$ there exists $M \in \mathbb{N}$ such that for all $i \geq M$, $\mathrm{d}(0, s_i) = |s_i| > R$. Then $s_i > 0$ for all $i \geq M$ (if not, then $s_i < 0$ for some $i \geq M$ and then $\mathrm{d}(n, s_i) > R$ for all $n \in \mathbb{N}$, a contradiction).

($\Leftarrow$) Suppose (s_i) is eventually positive. Then there exists $M_1 \in \mathbb{N}$ such that $s_i > 0$ for all $i \geq M_1$. Since (s_i) is coarse, it is a K-sequence for some $K > 0$. Also, since (s_i) is proper, it goes to infinity so there exists $M_2 > 0$ such that $\mathrm{d}(0, s_i) > K$ for all $i \geq M_2$. Set $M = \max\{M_1, M_2\}$.

First we show $\{n\}_{n \geq s_M} \subset \bigcup \bar{B}(s_i, K)$. Let $n \geq s_M$. Since $s_i \to \infty$, there is $t \geq M$ such that $\mathrm{d}(0, s_i) > n$ for all $i \geq t$. Then $s_i > n$ for all $i \geq t$. Let r be the smallest i such that $s_i > n$. We show $n \in \bar{B}(s_r, K)$. We know $s_{r-1} \leq n$. We also know $s_r - s_{r-1} \leq K$, so $s_r - K \leq s_{r-1}$. Then

$$s_r - K \leq s_{r-1} \leq n < s_r < s_r + K.$$

Thus $\{n\}_{n\geq s_M} \subset \bigcup \bar{B}(s_i,K)$. Also $\{0,1,2,\ldots,\lfloor s_M \rfloor\} \subset \bar{B}(0,\lfloor s_M \rfloor\})$ so the entire $\{n\} \subset \bigcup \bar{B}(s_i,\max\{K,\lfloor s_M \rfloor\})$.

For the other inclusion, set $R = \max\{|s_1|,|s_2|,\ldots,|s_{M_1-1}|\}$. Then we have $\{s_1,s_2,\ldots,s_{M_1-1}\} \subset \bar{B}(0,R)$. Also, $\{s_i\}_{i\geq M_1} \subset \bigcup \bar{B}(n,1)$ so $\{s_i\} \subset \bigcup \bar{B}(n,\max\{R,1\})$. ■

Theorem 3. $|\partial_H(\mathbb{R})| = 2$.

Proof. First we show that there are two distinct boundary points in $\partial_H(\mathbb{R})$, $[(n)]$ and $[(-n)]$. Suppose to the contrary that (n) is Hausdorff equivalent to $(-n)$. Then $\{n\} \subset \bigcup \bar{B}(-n,N)$ and $\{-n\} \subset \bigcup \bar{B}(n,N)$ for some $N \in \mathbb{N}$. Then $N+1 \in \bar{B}(-n,N)$ for some $n \in \mathbb{N}$. Then $\mathrm{d}(N+1,-n) = |N+1-(-n)| = N+1+n \leq N$, a contradiction.

Now we must show there are only two boundary points in $\partial_H(\mathbb{R})$. Let (s_i) be a coarse sequence in $\mathbb{R}$. We can assume $s_1 = 0$. We want to show that (s_i) can only be equivalent to either (n) or $(-n)$. Since (s_i) is coarse, it is a K-sequence for some $K > 0$. Also, since (s_i) is proper, it goes to infinity so there exists $M > 0$ such that $\mathrm{d}(0,s_i) > K$ for all $i \geq M$. We consider two cases.

Case 1: $s_M \geq 0$. We show $s_i > 0$ for all $i \geq M$; then $[(s_i)] = [(n)]$ by Proposition 1. We know $\mathrm{d}(0,s_M) > K$, which means

$$\begin{aligned} \mathrm{d}(0,s_M) = |s_M| &> K \\ s_M &> K \\ s_M - K &> 0. \end{aligned}$$

Since (s_i) is a K-sequence, we know

$$\begin{aligned} |s_{M+1} - s_M| &\leq K \\ -K \leq s_{M+1} - s_M &\leq K \\ 0 < s_M - K \leq s_{M+1} &\leq s_M + K. \end{aligned}$$

Thus $s_{M+1} > 0$. Therefore $s_i > 0$ for all $i \geq M$ by induction.

Case 2: $s_M < 0$. We can prove that (s_i) is Hausdorff equivalent to $(-n)$ if and only if (s_i) is eventually negative in a similar manner to how we proved Proposition 1. Then we can show (s_i) is Hausdorff equivalent to $(-n)$ in a way that is analogous to Case 1. ■

Now we show that the Hausdorff boundary is a coarse invariant. To that end we show that a coarse function induces a function on the boundary.

Lemma 1. *Suppose $f : X \to Y$ is a coarse function. Suppose $[(s_i)],[(t_i)] \in \partial_H(X)$ with $[(s_i)] = [(t_i)]$. Then $[(f(s_i))] = [(f(t_i))]$.*

Proof. Suppose $[(s_i)],[(t_i)] \in \partial_H(X)$ with $[(s_i)] = [(t_i)]$. Then there exists $R > 0$ such that $\{t_i\} \subset \bigcup \bar{B}(s_i,R)$ and $\{s_i\} \subset \bigcup \bar{B}(t_i,R)$. To show $[(f(s_i))] = [(f(t_i))]$, we find $R' > 0$ such that $\{f(t_i)\} \subset \bigcup \bar{B}(f(s_i),R')$ and $\{f(s_i)\} \subset \bigcup \bar{B}(f(t_i),R')$. Since f

is bornologous, there exists an $R' > 0$ such that if $\mathrm{d}(x,y) \leq R$, then $\mathrm{d}(f(x), f(y)) \leq R'$. Let $i \in \mathbb{N}$. Since $\{t_i\} \subset \bigcup \bar{B}(s_i, R)$, there exists $j \in \mathbb{N}$ such that $t_i \in \bar{B}(s_j, R)$. Then $\mathrm{d}(t_i, s_j) \leq R$ so $\mathrm{d}(f(t_i), f(s_j))$
$\leq R'$ and $f(t_i) \in \bar{B}(f(s_j), R')$. Symmetrically, we can use the fact that $\{s_i\} \subset \bigcup \bar{B}(t_i, R)$ to obtain $\{f(s_i)\} \subset \bigcup \bar{B}(f(t_i), R')$. Thus $(f(s_i))$ is Hausdorff equivalent to $(f(t_i))$. ■

Theorem 4. *If X and Y are coarsely equivalent, then $|\partial_H(X)| = |\partial_H(Y)|$.*

Proof. We have coarse functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identities. We define $f^* : \partial_H(X) \to \partial_H(Y)$ by setting $f^*([(s_i)]) = [(f(s_i))]$ for all $[(s_i)] \in \partial_H(X)$. From Lemma 1, we know f^* is well-defined. We want to show that f^* is bijective.

First, to show f^* is injective, suppose $[(s_i)], [(t_i)] \in \partial_H(X)$ and $(f(s_i))$ is Hausdorff equivalent to $(f(t_i))$. We show $[(s_i)] = [(t_i)]$. From Lemma 1, we obtain $[(g \circ f(s_i))] = [(g \circ f(t_i))]$. Since $g \circ f$ is close to the identity of X, there exists $K > 0$ such that $\mathrm{d}(g \circ f(s_i), s_i) \leq K$ for all $i \in \mathbb{N}$. Thus $\{s_i\} \subset \bigcup \bar{B}(g \circ f(s_i), K)$. Likewise, $\{g \circ f(s_i)\} \subset \bigcup \bar{B}(s_i, K)$. Thus $[(s_i)] = [(g \circ f(s_i))]$. Symmetrically $[(t_i)] = [(g \circ f(t_i))]$. Therefore $[(s_i)] = [(t_i)]$.

Next, to show f^* is surjective, let $[(s_i)] \in \partial_H(Y)$. Then $[g(s_i)] \in \partial_H(X)$. Similarly, $[f(g(s_i))] \in \partial_H(Y)$. Since $f \circ g$ is close to the identity, we have $\mathrm{d}(f(g(s_i)), s_i) \leq K$ for some $K > 0$. Therefore $\{f(g(s_i))\} \subset \bigcup \bar{B}(s_i, K)$ and $\{s_i\} \subset \bigcup \bar{B}(f(g(s_i)), K)$. Thus, $f^*([g(s_i)]) = [(s_i)]$. ■

From Theorem 3 we know $|\partial_H(\mathbb{R})| = 2$. A similar argument shows that $|\partial_H(\mathbb{R}^+)| = 1$. Therefore the real line and a ray are not coarsely equivalent since their Hausdorff sequential boundaries have different cardinalities.

Coarse sequences in the real plane

We characterize those coarse sequences in $\mathbb{R}^2$ that are Hausdorff equivalent to straight sequences. Given a slope $m \in \mathbb{R}$, the coarse sequence $((n, mn))$ goes to infinity along the line $y = mx$ in the positive x-direction.

Proposition 2. *Let $m \in \mathbb{R}$. A coarse sequence $((x_i, y_i))$ in $\mathbb{R}^2$ is Hausdorff equivalent to the sequence $((n, mn))$ if and only if it satisfies the following conditions.*

1. *There exists $M > 0$ such that $|y_i - mx_i| \leq M$ for all $i \in \mathbb{N}$.*
2. *The sequence (x_i) is Hausdorff equivalent to (n) in $\mathbb{R}$.*

Proof. We assume that $((x_i, y_i))$ starts at $(0,0)$.

($\Rightarrow$) First, suppose $((x_i, y_i))$ is Hausdorff equivalent to $((n, mn))$. Then there exists $R > 0$ such that $\{(x_i, y_i)\} \subset \bigcup \bar{B}((n, mn), R)$ and $\{(n, mn)\} \subset \bigcup \bar{B}((x_i, y_i), R)$.

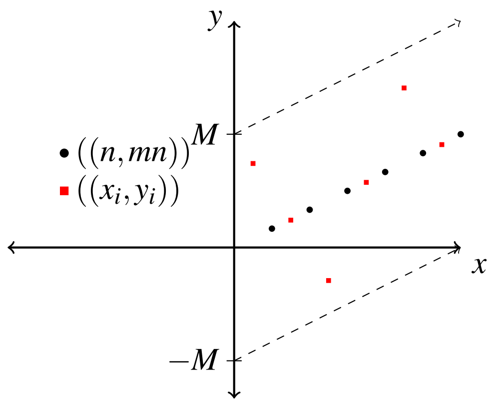

Figure 3: $((x_i,y_i))$ is Hausdorff equivalent to $((n,mn))$

Since $\{(x_i,y_i)\} \subset \bigcup \bar{B}((n,mn),R)$, given $i \in \mathbb{N}$, $(x_i,y_i) \in \bar{B}((k,mk),R)$ for some $k \in \mathbb{N}$ so $\mathrm{d}((x_i,y_i),(k,mk)) \le R$. Then

$$\begin{aligned}|y_i - mk| &= \sqrt{(y_i - mk)^2} \\ &\le \sqrt{(x_i-k)^2 + (y_i - mk)^2} \\ &\le R.\end{aligned}$$

Similarly, $|x_i - k| \le R$. Then

$$|mx_i - mk| = |m||x_i - k| \le |m|R$$

so

$$|y_i - mx_i| \le |y_i - mk| + |mk - mx_i| \le R + |m|R.$$

We use Proposition 1 to show (x_i) is Hausdorff equivalent to (n) in $\mathbb{R}$. Thus we need to show (x_i) is a coarse sequence and that it is eventually positive. We know that $((x_i,y_i))$ is a coarse sequence. That is, it is bornologous and proper. Since it is bornologous, there exists $K > 0$ such that $\mathrm{d}((x_i,y_i),(x_{i+1},y_{i+1})) \le K$ for all $i \in \mathbb{N}$. Then

$$\begin{aligned}\mathrm{d}(x_i,x_{i+1}) &= |x_i - x_{i+1}| \\ &= \sqrt{(x_{i+1}-x_i)^2} \\ &\le \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2} \\ &= \mathrm{d}((x_i,y_i),(x_{i+1},y_{i+1})) \le K\end{aligned}$$

for all $i \in \mathbb{N}$ so (x_i) is bornologous.

To show that (x_i) is proper, suppose to the contrary that there exists $T > 0$ such that for all $N \in \mathbb{N}$, there exists $i \ge N$ such that $|x_i| \le T$. In that case, $|y_i| \le |y_i - mx_i| + |mx_i| \le M + |m|T$ so

$$\mathrm{d}((0,0),(x_i,y_i)) \le |x_i| + |y_i| \le T + M + |m|T,$$

a contradiction since $(x_i, y_i) \to \infty$.

Now we show (x_i) is eventually positive. Recall that we know $\{(x_i, y_i)\} \subset \bigcup \bar{B}((n, mn), R)$. Since $((x_i, y_i))$ is proper, there exists $N > 0$ such that $\mathrm{d}((0,0),(x_i, y_i)) > R + R\sqrt{1+m^2} + 1$ for all $i \geq N$. Suppose to the contrary that $x_i \leq 0$ for some $i \geq N$. We show that $(x_i, y_i) \notin \bar{B}((n, mn), R)$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. We consider two cases on the size of n.

Case 1: $n > R$. Then

$$\mathrm{d}((x_i, y_i),(n, mn)) \geq |x_i - n| = n - x_i \geq n > R.$$

Case 2: $n \leq R$. We have

$$\mathrm{d}((x_i, y_i),(n, nm)) + \mathrm{d}((n, nm),(0,0)) \geq \mathrm{d}((x_i, y_i),(0,0)).$$

Then

$$\begin{aligned}
\mathrm{d}((x_i, y_i),(n, mn)) &\geq \mathrm{d}((x_i, y_i),(0,0)) - \mathrm{d}((n, mn),(0,0)) \\
&= \mathrm{d}((x_i, y_i),(0,0)) - n\sqrt{1+m^2} \\
&\geq \mathrm{d}((x_i, y_i),(0,0)) - R\sqrt{1+m^2} \\
&\geq R + R\sqrt{1+m^2} + 1 - R\sqrt{1+m^2} > R.
\end{aligned}$$

In both cases we have $(x_i, y_i) \notin \bar{B}((n, mn), R)$.

($\Leftarrow$) Next, we suppose that there exists an $M > 0$ such that $|y_i - mx_i| \leq M$ for all $i \in \mathbb{N}$ and (x_i) is Hausdorff equivalent to (n) in $\mathbb{R}$. We want to show $((x_i, y_i))$ is Hausdorff equivalent to $((n, mn))$.

First we show $\{(x_i, y_i)\} \subset \bigcup \bar{B}((n, mn), R)$ for some $R > 0$. By Proposition 1 we know that (x_i) is eventually positive, that is, there exists $T > 0$ such that $x_i > 0$ for all $i \geq T$. It suffices to show $\{(x_i, y_i)\}_{i \geq T} \subset \bigcup \bar{B}((n, mn), 1 + M + |m|)$. Suppose $i \geq T$. Then $|x_i - \lfloor x_i \rfloor| \leq 1$ so $|mx_i - m\lfloor x_i \rfloor| \leq |m|$ and

$$|y_i - m\lfloor x_i \rfloor| \leq |y_i - mx_i| + |mx_i - m\lfloor x \rfloor| \leq M + |m|$$

so

$$\begin{aligned}
\mathrm{d}((x_i, y_i),(\lfloor x \rfloor, m\lfloor x \rfloor)) &= \sqrt{(x_i - \lfloor x_i \rfloor)^2 + (y_i - m\lfloor x_i \rfloor)^2} \\
&\leq |x_i - \lfloor x_i \rfloor| + |y_i - m\lfloor x_i \rfloor| \\
&\leq 1 + M + |m|.
\end{aligned}$$

Now we show that $\{(n, mn)\} \subset \bigcup \overline{B}((x_i, y_i), R)$ for some $R > 0$. Let $n \in \mathbb{N}$. Since $\{x_i\}$ is Hausdorff equivalent to (n) there is $S > 0$ such that $\{n\} \subset \bigcup \overline{B}(x_i, S)$. Thus $|x_i - n| \leq S$ for some $i \in \mathbb{N}$. Now $|y_i - mx_i| \leq M$ and $|mx_i - mn| \leq |m|S$ so $|y_i - mn| \leq |y_i - mx_i| + |mx_i - mn| \leq M + |m|S$ and

$$\mathrm{d}((x_i, y_i),(n, mn)) \leq |x_i - n| + |y_i - mn| \leq S + M + |m|S.$$

■

A Partial Order on the Boundary

Our motivation for defining the Hausdorff sequential ends was to find a way to distinguish between the Euclidean plane and half plane. We know that the real line and ray are not coarsely equivalent. That was shown in [2] and again after Theorem 4 using the Hausdorff sequential boundary.

We encounter difficulties in the analog in two dimensions. Using the sequential ends from [2] we find that both the plane and the half plane have a single end so the invariant is inconclusive. Using the Hausdorff sequential boundary defined here we have the opposite issue; both the plane and half plane have infinitely many Hausdorff boundary points. One would guess that the cardinalities are also the same so again the invariant is inconclusive. One way to proceed is to put some structure on the boundary that is invariant under coarse equivalence. We do that here by defining a partial order on the boundary.

Recall that a relation $\leq$ on a set A is a partial order if it is reflexive, anti-symmetric, and transitive. The relation is reflexive if $a \leq a$ for all $a \in A$. It is anti-symmetric if $a \leq b$ and $b \leq a$ implies $a = b$. Finally, it is transitive if $a \leq b$ and $b \leq c$ implies $a \leq c$.

Definition 2. *Given two coarse sequences (s_i) and (t_i) in a metric space X, we say $[(s_i)] \leq [(t_i)]$ if there exists $R > 0$ such that $\{s_i\} \subset \bigcup \bar{B}(t_i, R)$.*

To show this definition gives a partial order, we need to show it is reflexive, anti-symmetric, and transitive. It is obvious from the definition that it is reflexive. To show the definition is anti-symmetric, let $[(s_i)], [(t_i)] \in \partial_H(X)$ with $[(s_i)] \leq [(t_i)]$ and $[(t_i)] \leq [(s_i)]$. This condition is exactly that of Definition 1 so $[(s_i)] = [(t_i)]$. Transitivity of this partial order is just half of the transitivity of Hausdorff equivalence that we proved after Definition 1.

It turns out that $\partial_H(\mathbb{R}^2)$ has a greatest element but no least element (later on we show that it has infinitely many minimal elements). Recall an element z of a partially ordered set A is the greatest element if $z \geq a$ for all $a \in A$. Note greatest elements are always unique. To get the greatest element of $\partial_H(\mathbb{R}^2)$ we define a sequence that spirals to fill out the entire integer lattice.

Definition 3. *Let (z_i) be the sequence in $\mathbb{R}^2$ that starts at $(0,0)$, then continues in a square:*

$$(1,0), (1,1), (0,1), (-1,1), (-1,0), (-1,-1), (0,-1), (1,-1), (2,-1).$$

Then we continue by going around in the next square by doing

$$\begin{gathered}(2,0), (2,1), (2,2), (1,2), (0,2), (-1,2), (-2,2), \\ (-2,1), (-2,0), (-2,-1), (-2,-2), \\ (-1,-2), (0,-2), (1,-2), (2,-2), (3,-2).\end{gathered}$$

This pattern is then continued for each subsequent square, as seen in Figure 4.

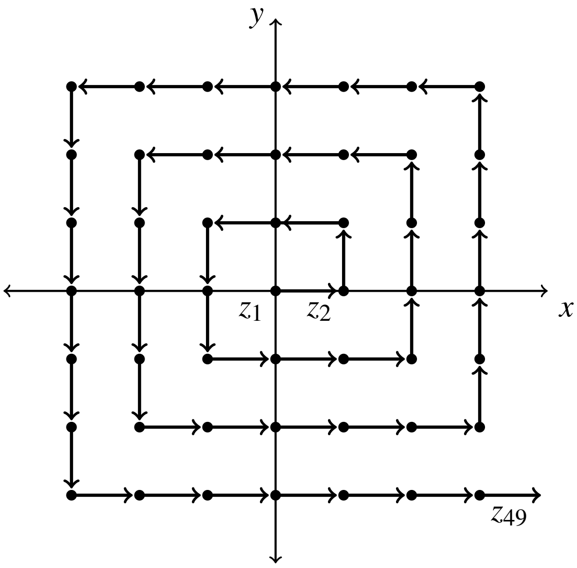

Figure 4: (z_i): The greatest element in $\mathbb{R}^2$

The spiral (z_i) gives the greatest element $[(z_i)]$ in $\partial_H(\mathbb{R}^2)$ since for any coarse sequence (s_i) in $\mathbb{R}^2$, $\{s_i\} \subset \mathbb{R}^2 = \bigcup \bar{B}(z_i, 1)$.

Proposition 3. *A sequence (t_i) is Hausdorff equivalent to (z_i) if and only if $\bigcup \bar{B}(t_i, R) = \mathbb{R}^2$ for some $R > 0$.*

Proof. ($\Rightarrow$) Suppose (t_i) is Hausdorff equivalent to (z_i). Then there exists $T > 0$ such that $\{t_i\} \subset \bigcup \bar{B}(z_i, T)$ and $\{z_i\} \subset \bigcup \bar{B}(t_i, T)$. We show $\bigcup \bar{B}(t_i, T+1) = \mathbb{R}^2$. Let $(x,y) \in \mathbb{R}^2$. Then $(x,y) \in \bar{B}(z_j, 1)$ for some $j \in \mathbb{N}$. We know $z_j \in \bar{B}(t_k, T)$ for some $k \in \mathbb{N}$. Thus $(x,y) \in \bar{B}(t_k, T+1)$.

($\Leftarrow$) Suppose $\bigcup \bar{B}(t_i, R) = \mathbb{R}^2$. We know $\bigcup \bar{B}(z_i, 1) = \mathbb{R}^2$. Thus $\{t_i\} \subset \bigcup \bar{B}(z_i, 1)$ and $\{z_i\} \subset \bigcup \bar{B}(t_i, R)$. ■

We now show that each $[(n, mn)] \in \partial_H(\mathbb{R}^2)$ is minimal. Note that by the characterization in Proposition 2 it is obvious that these elements are all distinct for distinct "slopes" m.

An element k of a partially ordered set A is a minimal element if $a < k$ is not true for all $a \in A$. In other words, k is minimal if for all $a \in A$, $a \leq k$ implies $a = k$.

Lemma 2. *If $[((x_i, y_i))] \leq [((n, mn))]$ in $\mathbb{R}^2$, then there exists an $M > 0$ such that $|y_i - mx_i| \leq M$ for all $i \in \mathbb{N}$ and (x_i) is Hausdorff equivalent to (n) in $\mathbb{R}$.*

Proof. Notice in the proof of Proposition 2, in the forward direction, that we only used the first inclusion $\{(x_i, y_i)\} \subset \bigcup \bar{B}((n, mn), R)$. ■

Proposition 4. *For all $m \in \mathbb{R}$, the boundary point $[((n, mn))]$ of $\mathbb{R}^2$ is minimal.*

Proof. We show that $[((x_i, y_i))] < [((n, mn))]$ cannot happen. Suppose $[((x_i, y_i))] \leq [((n, mn))]$. Then by Lemma 2 there is an $M > 0$ such that $|y_i - mx_i| \leq M$ for all $i \in \mathbb{N}$ and (x_i) is Hausdorff equivalent to (n). Then by Proposition 2 $[((x_i, y_i))] = [((n, mn))]$. ■

We believe there are many more minimal elements in $\partial_H(\mathbb{R}^2)$, including the one indicated in Figure 5.

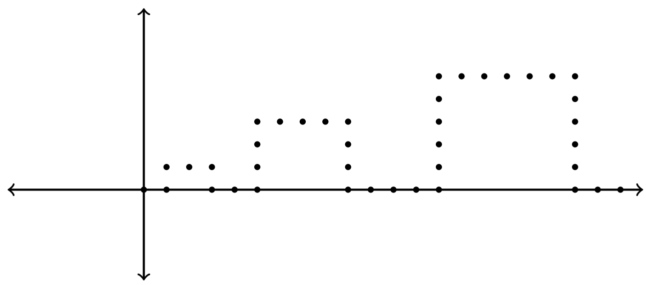

Figure 5: Another sequence giving a minimal element of $\partial_H(\mathbb{R}^2)$

Finally, we note that the partial order on the Hausdorff boundary is a coarse invariant. Suppose two coarse functions $f : X \to Y$ and $g : Y \to X$ form a coarse equivalence between X and Y. Then if $[(s_i)], [(t_i)] \in \partial_H(X)$ with $[(s_i)] \leq [(t_i)]$, then $f^*([(s_i)]) \leq f^*([(t_i)])$ by the argument in the proof of Lemma 1. Also, since $g \circ f$ is close to the identity, for all $[(s_i)] \in \partial_H(X)$, $g^* \circ f^*([(s_i)]) = [(s_i)]$. Thus g^* is the inverse of f^* and it follows that for all $[(s_i)], [(t_i)] \in \partial_H(X)$, $[(s_i)] \leq [(t_i)]$ if and only if $f^*([(s_i)]) \leq f^*([(t_i)])$. Thus $\partial_H(X)$ and $\partial_H(Y)$ are order isomorphic.

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The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before January 31, 2025. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall 2024 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 928 - 936

928. *Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*
Solve in the set of positive integers the following equation: $x^2+y^2 = 137(x-y)$.

929. *Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*
In how many ways can the rational $\frac{2025}{2024}$ be written as the product of two rational numbers of the form $\frac{(n+1)}{n}$, where n is a positive integer?

930. *Proposed by Mathew Cropper, Eastern Kentucky University, Richmond, KY.*
The Mycielski construction is done to a finite simple graph G producing a graph $M(G)$ as follows: set the vertex set of G to be $\{v_1, v_2, \ldots, v_k\}$, then add a set of vertices $\{u_1, u_2, \ldots, u_k\}$ and one more vertex w. Set u_i to be adjacent to every vertex in G to which v_i is adjacent and make w adjacent to every vertex in $\{u_1, u_2, \ldots, u_k\}$. Note that the set of vertices $\{u_1, u_2, \ldots, u_k\}$ is an independent set [Introduction to Graph Theory, West, pg. 205]. Let $M^n(G)$ denote the graph obtained from a given finite simple graph G by applying the Mycielski construction n times. Determine a formula for the number of edges in the graph $M^n(G)$.

931. *Proposed by Richard Hasenauer, Eastern Kentucky University, Richmond, KY.*
Prove that 7! divides $n^7 - 14n^5 + 49n^3 - 36^n$ for all positive integers n.

932. *Proposed by Tom Richmond, Western Kentucky University, Bowling Green, KY.*
If a and b are distinct square-free natural numbers and c and d are nonzero rational numbers, find necessary and sufficient conditions for $c\sqrt{a} + d\sqrt{b}$ to be a nonzero rational number.

933. *Proposed by Tom Richmond, Western Kentucky University, Bowling Green, KY.*
For cube-free integers $a, b > 1$ and nonzero rationals c, d, show that $c\sqrt[3]{a} + d\sqrt[3]{b}$ is rational if and only if $a = b$ and $c = -d$.

934. *Proposed by John Wilson, recently retired from Centre College, Danville, KY.*
A Squarely puzzle is a logic puzzle played on a 5×5 grid. The solution requires that the digits 1 through 9 be placed in the grid with two rules:

1. No digit appears more than once in any row, column or diagonal;
2. the 25 cells must contain exactly 3 copies of 8 of the digits and one copy of the ninth digit.

You are given the five digits of each row, column and diagonal.

Row	**Column**	**Diagonal**
13458	12459	
12369	34689	\12589
24569	35678	
24568	12569	\35689
13789	12348	

Get 10 free puzzles at squarelypuzzle.com.

935. *Proposed by the editor.*
The binomial transform of sequence $a_0, a_1, a_2, \ldots, a_n$ is sequence $b_0, b_1, b_2, \ldots, b_n$ where

$$b_k = \sum_{i=0}^{k} (-1)^i \left(\begin{array}{c} k \\ i \end{array} \right) a_i$$

Starting with sequence $a_0 = 1, a_1 = -2, a_2 = 4, a_3 = -8, a_4 = 16$, find b_4.

936. *Proposed by the editor.*
A pair of numbers (A, M) is called *amicable* when $\sigma(A) - A = M$ and $\sigma(M) - M = A$ where $\sigma(n)$ is the sum of all positive divisors of n. The smallest amicable pair is (220,284) which was known to Pythagoras.
4230338853909611459680566139419405 3 is one member of a previously-unknown amicable pair. Find the other member.

SOLUTIONS TO PROBLEMS 911 - 919

Problem 911. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Romania.*
Solve for real numbers:

$$\begin{cases} 2\sin x + 1 = 2\sin y + 2\sin z \\ (\sin x + \sin y - \sin z)^2 + (\sin x - \sin y + \sin z)^2 + 1 = \sin x + \sin y + \sin z. \end{cases}$$

Solution *by Brian Beasley, Simpsonville, SC.*

Let $a = \sin x, b = \sin y$, and $c = \sin z$. Then the first equation yields $c = a - b + \frac{1}{2}$, so substituting into the second equation produces

$$\left(2b - \frac{1}{2}\right)^2 + \left(2a - 2b + \frac{1}{2}\right)^2 + 1 = 2a + \frac{1}{2}.$$

This equation in turn is equivalent to $4a^2 + (-8b)a + (8b^2 - 4b + 1) = 0$. Then we obtain

$$a = b \pm \frac{\sqrt{-4b^2 + 4b - 1}}{2}.$$

Since $-4b^2 + 4b - 1 = -(2b-1)^2$ and a is real, we conclude that $b = \frac{1}{2}$ and thus $(\sin x, \sin y, \sin z) = (a, b, c) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Hence each of x, y, and z must be in the set

$$\left\{\frac{\pi}{6} + 2\pi\, n : n \in \mathbb{Z}\right\} \cup \left\{\frac{5\pi}{6} + 2\pi\, n : n \in \mathbb{Z}\right\} \cup \left\{\frac{\pi}{2} \pm \frac{\pi}{3} + 2\pi\, n : n \in \mathbb{Z}\right\}.$$

Also solved by Kee-Wai Lau, Hong Kong, China; and the proposer.

Problem 912. *Proposed by Mihaly Bencze, Brasov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Solve in real numbers the following equation:

$$x^2-5x-2\sqrt{x-2}+7+\log_2\frac{x^2-5x+8}{\sqrt{x-2}}+\log_3\frac{x^2-5x+8}{2\sqrt{x-2}}=0.$$

Solution *by Brian Beasley, Simpsonville, SC.*

For $x>2$, let $f(x)=x^2-5x-2\sqrt{x-2}+7$ and $g(x)=\frac{x^2-5x+8}{\sqrt{x-2}}$. Then

$$f'(x)=2x-5-\tfrac{1}{\sqrt{x-2}} \text{ and } g'(x)=\tfrac{(x-3)(3x-4)}{2(x-2)^{3/2}}.$$

Thus $g'(x)<0$ for $2<x<3$ and $g'(x)>0$ for $x>3$ so g is decreasing on (2,3) and increasing on $(3,\infty)$. Similarly, for $2<x<3$,

$$2x-5<1<\frac{1}{\sqrt{x-2}}$$

and for $x>3$,

$$2x-5>1>\frac{1}{\sqrt{x-2}},$$

so f is also decreasing on (2,3) and increasing on $(3,\infty)$. Since $p(x)=\log_2 x$ and $q(x)=\log_3 x$ are increasing on $(0,\infty)$, the function

$$h(x)=f(x)+\log_2 g(x)+\log_3\frac{g(x)}{2}$$

is decreasing on (2,3) and increasing on $(3,\infty)$. Finally, since $h(3)=0$, we conclude that the unique real solution of the given equation is $x=3$.

Also solved by Kee-Wai Lau, Hong Kong, China; and the proposers.

Problem 913. *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Find $\lim\limits_{n\to\infty}\left(\left(\frac{\pi^2}{6}-\sum\limits_{k=1}^{n}\frac{1}{k^2}\right)*e^{x_n}\right)$ where $x_n=\sum\limits_{k=1}^{n}\frac{1}{k}$.

Solution *by Seán Stewart, King Abdullah University, Saudi Arabia.*

Denote the limit to be found by L. Recalling the definitions for the n^{th} harmonic number $H_n=\sum_{k=1}^n\frac{1}{k}$ and the generalized n^{th} harmonic number of order two $H_n^{(2)}=\sum_{k=1}^n\frac{1}{k^2}$, in terms of these two numbers the limit can be expressed as

$$=\lim_{n\to\infty}\left[\left(\frac{\pi^2}{6}-H_n^{(2)}\right)e^{H_n}\right].$$

From the asymptotic expansions for H_n and $H_n^{(2)}$ as $n \to \infty$, namely

$$Hn = \log(n) + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O\left(\frac{1}{n^4}\right),$$

and

$$H_n^{(2)} = \frac{\pi^2}{6} - \frac{1}{n} + \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right),$$

where γ denotes the Euler–Mascheroni constant, for the exponential term in the limit we have

$$\begin{aligned} e^{H_n} &= exp\left[\log(n) + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O\left(\frac{1}{n^4}\right)\right] \\ &= ne^{\gamma} exp\left[\frac{1}{2n} - \frac{1}{12n^2} + O\left(\frac{1}{n^4}\right)\right] \\ &= ne^{\gamma}\left[1 + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right]. \end{aligned}$$

The limit may therefore be written as

$$\begin{aligned} L &= \lim_{n\to\infty}\left[\left\{\frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)\right\} * ne^{\gamma}\left\{1 + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right\}\right] \\ &= e^{\gamma}\lim_{n\to\infty}\left[\left\{1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right\} * \left\{1 + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right\}\right] \\ &= e^{\gamma}\lim_{n\to\infty}\left[1 + O\left(\frac{1}{n^2}\right)\right] = e^{\gamma}, \end{aligned}$$

which is the required value for the limit.

Also solved by Kee-Wai Lau, Hong Kong, China; Henry Ricardo, Westchester Area Math Circle, Purchase, NY; and the proposers.

Problem 914. *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Compute $\lim\limits_{n\to\infty} n\sqrt[n]{(2n-1)!!F_n}\sin\frac{1}{n^2}$ where F_n is the n^{th} Fibonacci number.

Solution *by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.*

It is well-known that $\lim\limits_{n\to\infty}\sqrt[n]{F_n} = \lim\limits_{n\to\infty}\frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$ and $n^2\sin\frac{1}{n^2} = \frac{\left(\sin\frac{1}{n^2}\right)}{(1/n^2)} \to 1$ as $n \to \infty$. Now we show that $\frac{\sqrt[n]{(2n-1)!!}}{n} \to \frac{2}{e}$ as $n \to \infty$:

$$(2n-1)!! = \frac{2n!}{2^n n!} \Rightarrow (2n-1)!! \sim \frac{2^n n^{n+1/2}}{e^n} \text{ by Stirling}$$
$$\Rightarrow \sqrt[n]{(2n-1)!!} \sim \frac{2n}{e}$$
$$\Rightarrow \frac{\sqrt[n]{(2n-1)!!}}{n} \sim \frac{2}{e}.$$

Therefore,

$$\sqrt[n]{(2n-1)!!F_n} sin \frac{1}{n^2} = \frac{\sqrt[n]{(2n-1)!!}}{n} * \sqrt[n]{F_n} * \frac{\left(\sin\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}$$
$$\to \frac{2}{e} * \frac{1+\sqrt{5}}{2}$$
$$= \frac{1+\sqrt{5}}{e} = 1.1905.$$

Also solved by Kee-Wai Lau, Hong Kong, China; Seán Stewart, King Abdullah University, Saudi Arabia; and the proposers.

Problem 915. *Proposed by Toyesh Prakash Sharma (student), Agra College, Agra, India.*
Evaluate the following integral:

$$\int \ln(1+x)\cdot(e^x + \frac{1}{e^x})dx + \int \frac{1}{x+1}\cdot(e^x - \frac{1}{e^x})dx.$$

Solution *by Daniel Vacaru, National Economic College, Piteşti, Romania.*

We write

$$\int \ln(1+x)\left(e^x + \frac{1}{e^x}\right)dx + \int \frac{1}{x+1}\left(e^x - \frac{1}{e^x}\right)dx$$
$$= \int \ln(1+x)\left(e^x + \frac{1}{e^x}\right)dx + \int [\ln(1+x)]'\left(e^x - \frac{1}{e^x}\right)dx$$
$$= \int \ln(1+x)\left(e^x + \frac{1}{e^x}\right)dx + \ln(x+1)\left(e^x - \frac{1}{e^x}\right)$$
$$- \int [\ln(1+x)]\left(e^x - \frac{1}{e^x}\right)' dx$$

$$= \int \ln(1+x)\left(e^x + \frac{1}{e^x}\right)dx + \ln(x+1)\left(e^x - \frac{1}{e^x}\right) - \int [\ln(1+x)]\left(e^x + \frac{1}{e^x}\right)dx$$

$$= \ln(x+1)\left(e^x - \frac{1}{e^x}\right) + C.$$

Also solved by Brian Beasley, Simpsonville, SC; Alexia Lorenzo and Diego Velazco (students) and Angel Plaza, Universidad de Lad Palmas de Gran Canaria, Spain; Henry Ricardo, Westchester Area Math Club, Purchase, NY; Etisha Sharma, Agra College, Agra, India; Seán Stewart, King Abdullah University, Saudi Arabia; and the proposer. [Note: Mathematica will compute this integral.]

Problem 916. *Proposed by Raluca Maria Caraion and Forică Anastase, "Alexandru Odobescu" High School, Lehliu-Gară, Călăraşi, Romania.*
Find: $\Omega = \lim_{p\to\infty} \frac{1}{p^a} \cdot \sum_{m=1}^{p} \sum_{n=1}^{m} \sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2}$.

Solution *by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.*

First we note the symmetry:

$$\sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2} = \sum_{k=0}^{n} \frac{k^2}{2k^2-2nk+n^2} = \sum_{k=0}^{n} \frac{(n-k)^2}{2k^2-2nk+n^2}$$

which yields

$$\sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2} = \frac{1}{2}\sum_{k=0}^{n} \frac{k^2+(n-k)^2}{k^2+(n-k)^2} = \frac{n+1}{2}.$$

Then using familiar formulas for $\sum j$ and $\sum j^2$, we obtain

$$\sum_{n=1}^{m} \sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2} = \frac{1}{2}\sum_{n=1}^{m}(n+1) = \frac{m(m+3)}{4}$$

and

$$\begin{aligned}\sum_{m=1}^{p} \sum_{n=1}^{m} \sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2} &= \frac{1}{4}\sum_{m=1}^{p}(m^2+3m) \\ &= \frac{1}{4}\left(\frac{p(p+1)(2p+1)}{6} + \frac{3p(p+1)}{2}\right) \\ &= \frac{p(p+1)(p+5)}{12}.\end{aligned}$$

It follows that
$$\Omega = \lim_{p\to\infty} \frac{p^{3-a}}{12}\left(1+\frac{6}{p}+\frac{5}{p^2}\right).$$
This is ∞ for $a<3$, $\frac{1}{12}$ for $a=3$, and 0 for $a>3$.

Also solved by Seán Stewart, King Abdullah University, Saudi Arabia; and the proposers.

Problem 917. *Proposed by Marian Ursărescu, "Roman Voda" College, Roman, Neamt, Romania, and Florică Anastase, "Alexandru Odobescu" High School, Lehliu-Gară, Călăraşi, Romania.*
Let $(a_n)_{n\geq 1}, (b_n)_{n\geqslant 1}$ be two sequences of real numbers defined by
$$a_n = \int\limits_1^n \left[\frac{n^2}{x}\right] dx;\ b_1>1,\ b_{n+1} = 1+\log(b_n)$$
where [*] denotes the greatest integer function. Find $L = \lim\limits_{n\to\infty} \frac{a_n\cdot \log \sqrt[n]{b_n}}{\log n^n}$.

Solution *by Albert Stadler, Herrliberg, Switzerland.*

$$\begin{aligned}\int_2^n \left\lfloor\frac{n^2}{x}\right\rfloor dx &= \sum_{m=n}^{n^2-1}\int_{\frac{n^2}{m+1}}^{\frac{n^2}{m}} \left\lfloor\frac{n^2}{x}\right\rfloor dx\\ &= \sum_{m=n}^{n^2-1} m\left(\frac{n^2}{m}-\frac{n^2}{m+1}\right)\\ &= n^2\sum_{m=n}^{n^2-1}\frac{1}{m+1} = n^2\left(H_{n^2}-H_n\right),\end{aligned}$$

where H_n is the n^{th} harmonic number. It is well-known that the asymptotic of the harmonic numbers is $H_n = \log n+\ \gamma+\ O\left(\frac{1}{n}\right), n\to\infty$. Hence
$$a_n =\ n^2\log n + O(n), n\to\infty.$$

Let $c_n = \log b_n$. Then $c_1>0$ and $c_{n+1} = \log(1+c_n)\leq c_n$. So (c_n) is a monotonically decreasing sequence of positive numbers which tends to a limit $c\geq 0$. That limit equals 0, for
$$c = \lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty}\log(1+c_n) = \log(1+c)$$
implies $c=0$. We have
$$\frac{1}{2}-\frac{x}{12}\leq \frac{x-\log(1+x)}{x\log(1+x)}\leq\frac{1}{2},\ \ x\leq> 0.$$

To prove these two inequalities we replace x by $e^y - 1$ and use Taylor's expansion of the exponential function. We then see that

$$\begin{aligned} x - &\log(1+x) - x\log(1+x)(1/2 - x/12) \\ &= e^y - 1 - y - \frac{7}{12}y\,(e^y - 1) + \frac{1}{12}y(e^{2y} - e^y) \\ &= -1 + e^y - \frac{5y}{12} - \frac{2e^y y}{3} + \frac{1}{12}e^{2y}y \\ &= \sum_{k=4}^{\infty}\left(1 + \frac{(2^{k-1} - 8)k}{12}\right) \geq 0 \end{aligned}$$

and

$$\begin{aligned} x\log(1+x) - 2\,(x - \log(1+x)) &= y\,(e^y - 1 - y) \\ &= 2 + y - 2e^y + ye^y \\ &= \sum_{k=3}^{\infty}\frac{y^k}{k!}(-2+k) \geqslant 0. \end{aligned}$$

Hence

$$\frac{1}{2} - \frac{c_k}{12} \leq \frac{c_k - \log(1+c_k)}{c_k\log(1+c_k)} = \frac{c_k - c_{k+1}}{c_k c_{k+1}} = \frac{1}{c_{k+1}} - \frac{1}{c_k} \leq \frac{1}{2}.$$

We sum over k from $k = 1$ to $k = n-1$ and get

$$\frac{1}{2}(n-1) - \frac{1}{12}\sum_{k=1}^{n-1} c_k \leq \frac{1}{c_n} - \frac{1}{c_1} \leq \frac{1}{2}(n-1)$$

which is equivalent to

$$\frac{1}{\frac{n}{2} - \frac{1}{2} + \frac{1}{c_1}} \leq \mathrm{c}_n \leq \frac{1}{\frac{n}{2} - \frac{1}{2} + \frac{1}{c_1} - \frac{1}{12}\sum\limits_{k=1}^{n-1} c_k}$$

However $\sum_{k=1}^{n-1} c_k = o(n)$, since c_n tends to 0. Thus $c_n = \frac{2}{n}(1 + o(1))$ and we conclude that

$$\lim_{n\to\infty}\frac{a_n\log(\sqrt[n]{b_n})}{\log n} = \lim_{n\to\infty}\frac{a_n c_n}{n\log n} = \frac{(n^2\log n + O(n))(\frac{2}{n})(1 + o(1))}{n\log n} = 2.$$

Also solved by Kee-Wai Lau, Hong Kong, China; Henry Ricardo, Westchester Area Math Circle, Purchase, NY; and the proposers. [It was pointed out that this problem appeared as #5728 in School Science and Mathematics Journal.]

Problem 918. *Proposed by Seán Stewart, King Abdullah University of Science and Technology, Saudi Arabia.*
If $k > 0$, evaluate $\int\limits_0^1 \frac{\log(1+x^k+x^{2k})}{x}dx$.

Solution *by Kee-Wai Lau, Hong Kong, China*

Denote the integral by I. We show that $I = \frac{\pi^2}{9k}$. By substituting $x = y^{1/k}$, we obtain

$$I = \frac{1}{k}\int_0^1 \frac{\log(1+y+y^2)}{y}dy = \frac{1}{k}(I_1 - I_2),$$

where

$$I_1 = \int_0^1 \frac{\log\left(1-y^3\right)}{y}dy \text{ and } I_2 = \int_0^1 \frac{\log\left(1-y\right)}{y}dy$$

Substituting $y = z^{\frac{1}{3}}$ in I_1, we obtain $I_1 = \frac{1}{3}I_2$. Hence $I = -\frac{2}{3k}I_2$. It is well known that $I_2 = -\frac{\pi^2}{6}$. This gives the result.

Also solved by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; and the proposer.

Problem 919. *Proposed by the editor*
Find the error in the following proof: We want to find $\lim\limits_{n\to\infty} \frac{4^n}{3^n}$. This is an $\frac{\infty}{\infty}$ form so we can apply L'Hopital's Rule. Let $L = \lim\limits_{n\to\infty} \frac{4^n}{3^n}$. Then

$$\begin{aligned} L = \lim_{n\to\infty}\frac{4^n}{3^n} &= \lim_{n\to\infty}\frac{4^n\cdot\ln 4}{3^n\cdot\ln 3} \qquad \text{by L'Hopital's Rule} \\ &= \lim_{n\to\infty}\frac{4^n}{3^n}\lim_{n\to\infty}\frac{\ln 4}{\ln 3} = L\cdot\frac{\ln 4}{\ln 3}. \end{aligned}$$

Subtracting L from both sides gives, $0 = L\cdot(\frac{\ln 4}{\ln 3} - 1)$ but $\frac{\ln 4}{\ln 3} - 1$ is not 0. Therefore $L = 0$.

Solution *by Carl Libis, Columbia Southern University, Orange Beach, AL.*

L is infinity and subtracting infinity from infinity does not give 0, This is the same error as

$2*\infty = \infty$

$2*\infty - 1*\infty = \infty - \infty = 0$

$(2-1)*\infty = 0$

Since 2-1 is not 0, $\infty = 0$

Other suggested errors contributed by by Brian Beasley, Simpsonville, SC; Henry Ricardo, Westchester Area Math Circle, Purchase, NY; and Seán Stewart, King Abdullah University, Saudi Arabia.

Kappa Mu Epsilon News

Edited by Mark Hughes, Historian
Updated information as of March 2024

News of chapter activities and other noteworthy KME events should be sent to

Mark Hughes, KME Historian
Frostburg State University
Department of Mathematics
Frostburg, MD 21532
or to
mhughes@frostburg.edu

Chapter News

AL Theta – Jacksonville State University
Chapter President – Nicholas Covalsen; 339 Total Members
Other Fall 2023 Officers: Adam Parton, Vice President; Jacob Skipper, Secretary; Lucas Saone, Treasurer; Dr. David Dempsey, Corresponding Secretary; and Dr. Jason Cleveland, Faculty Sponsor.
The Alabama Theta chapter met at least monthly in person during Fall 2023 and elected new officers during the first meeting in September. Highlights included a trivia night in October (prepared by our officers) and an ice cream social to relax after the last day of classes in December. We look forward to our spring initiation ceremony on March 8 and hopefully travelling to a regional convention.

AR Beta – Henderson State University
Chapter President – Alex Hunter; 74 Total MembersValerie Grigar, Treasurer
Other Fall 2023 Officers: Kristen Harper, Vice President; Trenton Moore, Secretary; Valerie Grigar, Treasurer; Catherine Leach, Corresponding Secretary and Faculty Sponsor.

CT Beta – Eastern Connecticut State University
Corresponding Secretary and Faculty Sponsor – Dr. Mehdi Khorami; 553 Total Members

CT Gamma – Central Connecticut State University
Corresponding Secretary – Gurbakhshash Singh ; 78 Total Members
Other Fall 2023 Officer: Nelson Castaneda, Faculty Sponsor.

GA Zeta – Georgia Gwinnett College
Chapter President – Edgar Derricho; 69 Total Members

Other Fall 2023 Officers: Gabriel Amat, Vice President; Matt Elenteny, Secretary; Dr. Jamye Curry Savage, Corresponding Secretary and Faculty Sponsor; and Dr. Livy Uko, Faculty Sponsor.

GA Theta – College of Coastal Georgia

Chapter President – Justin Von Gartzen; 31 Total Members; 4 New Members
Other Fall 2023 Officers: Andrea Olvera, Vice President; Zach Atkinson, Secretary; Casey Griffin, Treasurer; Aaron Yeager, Corresponding Secretary and Faculty Sponsor.
Over the Fall Semester the Georgia Theta Chapter of Kappa Mu Epsilon inducted four new members and we had two meetings.
New Initiates –Ansley Simpson, Faith Highland, Noah English, and Alexander Salgado.

IA Alpha – University of Northern Iowa

Chapter President – Grace Croat; 1119 Total Members; 6 New Members
Other Fall 2023 Officers: Quinn Robinson, Vice President; Krista Zimmer, Secretary; Rachel Wohlgemuth, Treasurer; and Dr. Mark D. Ecker, Corresponding Secretary and Faculty Sponsor.
Twelve student members of KME and three faculty members met on Wednesday, December 6, 2023 in Wright Hall for our Fall KME meeting/banquet. Isabel Harms presented her senior seminar project entitled "An Analysis of the Meteorological Effect on PM 2.5 Concentrations in Davenport, Iowa" and six new student members were initiated at our meeting.

IA Gamma – Morningside University

Chapter President – Isaiah Hinners; 445 Total Members
Other Fall 2023 Officers: Kelsey Schieffer, Vice President and Secretary; Fred Lageschulte, Treasurer; and Dr. Eric Canning, Corresponding Secretary and Faculty Sponsor.
There were no new initiates in Fall 2023, as we plan on having an initiation ceremony this Spring. Our KME math club met 6 different evenings during the Fall semester. At these meetings, we had one guest speaker, we watched the movie Jerry & Marge Go Large, designed a math club t-shirt, carved pumpkins for Halloween, and had a couple of game and pizza nights.

IL Zeta – Dominican University

Corresponding Secretary and Faculty Sponsor – Mihaela Blanariu; 461 Total Members
We have not initiated any new members in fall 2023.

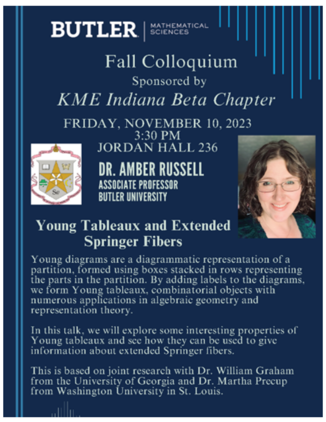
IL Theta – Benedictine University

Corresponding Secretary and Faculty Sponsor – Manmohan Kaur; 302 Total Members

IL Kappa – Aurora University
Chapter Presidents – Daniel Chacon, Samantha Zielinski; 97 Total Members
Other Fall 2023 Officers: Ethan Odean, Vice President; Brigid Redmond-Mattucci, Secretary; Emily Pastor, Treasurer; Lindsey Hill, Corresponding Secretary and Faculty Sponsor.
This fall, Illinois Kappa hosted a faculty panel, a board game event, and an escape room event. At the end of the semester, our chapter and the Department of Mathematics held two final exam student sessions for all mathematics students at Aurora University.

IN Alpha – Manchester University
Chapter President – Jonah Richards; 575 Total Members
Other Fall 2023 Officers: Morgan Chupp, Vice President; Michael DeBartolo, Secretary; Zach Hood, Treasurer; Tim Brauch, Corresponding Secretary and Faculty Sponsor.
Due to some sabbaticals, retirements, and covid, our club faded for a few years. However, we have a strong group of sophomore students who are very excited to revive the club. Our first meeting of the resurrected club is happening in mid-February. While our numbers are currently small, we are experiencing renewed interest in majoring (or at least minoring) in mathematics. For our first semester back, the focus is awareness and growth. We are going to find who is eligible for membership in KME and have some initiation ceremonies. Our current officers meet the criteria for initiation but haven't been inducted yet.

IN Beta – Butler University
Chapter President – Evan Blom; 452 Total Members
Other Fall 2023 Officers: Jenna Lane, Vice President; Dylan Laudenschlager, Sarah Moore, Secretaries; Rasitha Jayasekare, Corresponding Secretary and Faculty Sponsor.



KME Indiana Beta Chapter (KME chapter of Butler University, Indianapolis, IN) sponsored one of their department colloquia in the fall. The speaker was our very own faculty member, Dr. Amber Russell, spoke on "Young Tableaux and Extended Springer Fibers." A big shout out to our KME Indiana Beta Chapter President Evan Blom, Secretaries Dylan Laudenschlager and Sarah Moore, and VP Janna Lane for their work for organizing and hosting the event. The colloquium poster and a picture of the

KME club officers with the speaker follow.

From left to right: Evan Blom (president), Dr. Amber Russell, Jenna Lane (VP), Dylan Laudenschlager (secretary).

KS Alpha – Pittsburg State University
Chapter President – Daniel Crissinger; 2170 Total Members; 5 New Members
Other Fall 2023 Officers: Jonas Garibay, Vice President; Palak Chaudhary, Secretary; Dharani Maddi, Treasurer; Tim Flood, Corresponding Secretary; and Scott Thuong, Faculty Sponsor.
The Kansas Alpha section continued to hold monthly meetings. In addition to hosting student presentations, the club also held fun events like math trivia and playing the card game SET.

KS Beta – Emporia State University
Chapter President – Joe Rose; 1546 Total Members; 2 New Members
Other Fall 2023 Officers: Julia Whitaker, Vice President; Sky Willis, Secretary; Maliki Mosher, Treasurer; Tom Mahoney, Corresponding Secretary; and Brian Hollenbeck, Faculty Sponsor.

MD Delta – Frostburg State University
Chapter President – Kaitlyn Custer; 546 Total Members
Other Fall 2023 Officers: Ricky Day, Vice President; Faith James Sergent, Secretary; Dawson Hormuth, Treasurer; Mark Hughes, Corresponding Secretary and Faculty Sponsor; and Frank Barnet, Faculty Sponsor.
We had monthly meetings during the semester where we enjoyed puzzles, math

videos, and pizza. We also represented the Mathematics Department at our university's annual Majors Fair.

MI Beta – Central Michigan University
Chapter President – Maleia Thompson; 1762 Total Members
Other Fall 2023 Officers: Julia Savage, Vice President; Matt Sonnenschein, Secretary; Elijah Hayes, Treasurer; and Dmitry Zakharov, Corresponding Secretary and Faculty Sponsor.
In the fall semester, the Michigan Beta Chapter held six general meetings, a book sale, and a volunteer tutoring event. The chapter hosted three talks this semester. One was by Dr. Sivaram Narayan on proofs, and another was by Dr. Yeonhyang Kim and a graduate student on K-NN regressions in brain imaging. The last talk was by Dr. Chin-Yi Jean Chan on common zeros of polynomials. Other meetings included math bingo, math jeopardy, and other math related games.
New Initiates

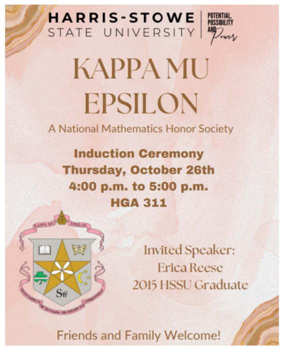
MO Epsilon – Central Methodist University
Chapter President – Jayklin Smith; 468 Total Members; 7 New Members
Other Fall 2023 Officers: Dillan Kessing, Vice President; Lydia Elder, Secretary; Ashlee Flowers, Treasurer; and Pam Gordy, Corresponding Secretary and Faculty Sponsor.

MO Theta – Evangel University
Chapter President – Jack Lin; 305 Total Members
Other Fall 2023 Officers: Ericsson McDermott, Vice President; and Dianne Twigger, Corresponding Secretary; and Jeremy Osborne, Faculty Sponsor.
The Missouri Theta KME chapter met 4 times during the Fall 2023 academic term. Many students also attended the MAKO conference held by Missouri State University in November.

MO Kappa – Drury Universityl
Chapter President – Samuel Fullbright; 345 Total Members
Other Fall 2023 Officers: Nicolette Gaston, Vice President; Hannah Ritter, Secretary; Kylie Warden, Treasurer; and Colin T. Baker, Corresponding Secretary and Faculty Sponsor.
Our chapter now holds weekly meets and is currently exploring the following topics: group theory and music, provable statements in the game of Hex, multidimensional tic-tac-toe rules, modeling chemical interactions using computer graphics.

MO Mu – Harris-Stowe State University
Corresponding Secretary and Faculty Sponsor – Ann Podleski; 117 Total Members; 9 New Members

Our Fall 2023 induction ceremony was a highlight for many and included remarks from a 2015 alum of HSSU and member of Kappa Mu Epsilon. (See flyer and photo.)
New Initiates – Jared Colbert, Jake Davis, Yasmina Dressen, Annika Fischer, Amani Griffin, Andrew Lawrence Haines, Morris Harris III, Jasmin Kraus, and Shayne Murphy.



MO Mu induction

NE Alpha – Wayne State College
Chapter President – Morgan Mitchell; 1069 Total Members
Other Fall 2023 Officers: Madi Minnehan, Vice President; Lacey Cruise, Secretary; Lily Shafer, Treasurer; and Dr. Jenny Langdon, Corresponding Secretary and Faculty Sponsor.
The Nebraska Alpha chapter of KME met in conjunction with the campus Math Club every two weeks. The members entered the homecoming banner competition, hosted a speaker, and held a holiday party. Four new members will be initiated in January.

NE Beta – University of Nebraska Kearney
Corresponding Secretary and Faculty Sponsor – Dr. Katherine Kime; 939 Total Members
Samantha Bowland, initiated Spring 2023, sends her news: "This semester I'm student teaching in a local high school. I'm helping students with pre-calculus, geometry, and algebra 2. I'll be graduating in May with a degree in 6-12 Math Education."
Brooke Carlson, who graduated from UNK in Spring 2023, is now pursuing a master's degree in Mechanical and Materials Engineering, at the University of Nebraska Lincoln, with an emphasis in Manufacturing Engineering. Her current project concerns "bonding dissimilar materials to increase heat dissipation in electronic applications" and measuring the bond strengths in the samples she has created.

NY Nu – Hartwick College
Chapter President – Runyararo Chaora; 357 Total Members
Other Fall 2023 Officers: Dereck Cupernall, Vice President; Jake Thorry, Secretary; Liam Kinnane, Treasurer; and Dr. Min Chung, Corresponding Secretary and Faculty Sponsor.

NY Xi – Buffalo State University
Corresponding Secretary and Faculty Sponsor – Jane Cushman; 56 Total Members; 4 New Members

OK Alpha – Northeastern State University (Spring 2023)
Chapter President – Cade Clickenbeard; 1875 Total Members; 3 New Members
Other Spring 2023 Officers: Parker Childers, Vice President; Mark Buckles, Secretary, Treasurer, Corresponding Secretary, and Faculty Sponsor.
We did an ice cream social in the spring semester of 2023 and used Zoom to connect both the Broken Arrow and Tahlequah campuses so that students and faculty could communicate across both campuses.

OK Alpha – Northeastern State University
Chapter President – Ryan McAbee; 1880 Total Members; 5 New Members
Other Fall 2023 Officers: Allen Ortiz, Vice President; Mark Buckles, Secretary, Treasurer, Corresponding Secretary, and Faculty Sponsor.
We held an initiation ceremony on October 26, 2023 using Zoom to connect students at our Broken Arrow campus and our Tahlequah campus. Pizza and soft drinks were provided on both campuses.

PA Pi – Slippery Rock University
Chapter President – Spencer Kahley; 145 Total Members
Other Fall 2023 Officers: Boris Brimkov, Corresponding Secretary; and Amanda Goodrick, Faculty Sponsor.

We did not have any activities in Fall 2023, but we talked to students to recruit them as new members for next semester.

PA Rho – Thiel College
Chapter President – Steven Wright; 148 Total Members
Other Fall 2023 Officers: Emmalee Sheeler, Vice President; Juliana Peace, Secretary; Bailey Stilts, Treasurer; Dr. Russell Richins, Corresponding Secretary; and Dr. Jie Wu, Faculty Sponsor.
We had several meetings to plan activities and select our chapter t-shirts this year. Our main activity was the Challenge 24 contest and fundraiser to benefit the local food bank.

RI Beta – Bryant University
Corresponding Secretary – Prof. John Quinn; 217 Total Members
Other Fall 2023 Officer: Prof. Gao Niu, Faculty Sponsor.
We have our KME nominations and initiation ceremony every spring semester and we are planning to do the same for spring 2024.

TX Mu – Schreiner University
Chapter President – Jake Plummer; 209 Total Members
Other Fall 2023 Officers: Dom Civello, Vice President; Chris Jones, Secretary; Rachel Lynn, Corresponding Secretary and Faculty Sponsor.
The Texas Mu Chapter was able to meet in the fall of 2023 for lunch and board games.

WV Alpha – Bethany College
RCorresponding Secretary and Faculty Sponsor – Adam C. Fletcher; 198 Total Members
West Virginia Alpha has had a rather quiet fall semester. After our seniors graduated last spring, our membership numbers dropped. We are, however, looking forward hopefully to this spring's initiation ceremony. The chapter and our local Mathematics and Computer Science Club attended a handful of conferences virtually and participated in campus clean-ups and service projects. They are eagerly anticipating and planning for the annual Math/Science Day Competition for local high school students on campus in February and providing service to local high school mathematics competitions in the community.

Active Chapters of Kappa Mu Epsilon

Listed by date of installation

Chapter	Location	Installation Date
OK Alpha	Northeastern State University, Tahlequah	18 Apr 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
AL Alpha	Athens State University, Athens	5 Mar 1935
NM Alpha	University of New Mexico, Albuquerque	28 Mar 1935
IL Beta	Eastern Illinois University, Charleston	11 Apr 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 Apr 1937
OH Alpha	Bowling Green State University, Bowling Green	24 Apr 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	University of Central Missouri, Warrensburg	10 Jun 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 Jun 1941
MI Beta	Central Michigan University, Mount Pleasant	25 Apr 1942
NJ Beta	Montclair State University, Upper Montclair	21 Apr 1944
IL Delta	University of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 Mar 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 Jun 1947
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 Apr 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	University of Nebraska—Kearney, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960
OH Epsilon	Marietta College, Marietta	29 Oct 1960
MO Zeta	University of Missouri—Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood University, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 Apr 1965
AL Epsilon	Huntingdon College, Montgomery	15 Apr 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
TN Gamma	Union University, Jackson	24 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	McDaniel College, Westminster	30 May 1965
IL Zeta	Dominican University, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Truman State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel University, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 Mar 1971
KY Alpha	Eastern Kentucky University, Richmond	27 Mar 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 Apr 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973

NY Kappa	Pace University, New York	24 Apr 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State University, Joplin	8 May 1975
GA Alpha	State University of West Georgia, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin—Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sep 1978
IL Theta	Benedictine University, Lisle	18 May 1979
PA Mu	St. Francis University, Loretto	14 Sep 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C.W. Post Campus of Long Island University, Brookville	2 May 1983
MO Kappa	Drury University, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 Mar 1985
NE Delta	Nebraska Wesleyan University, Lincoln	18 Apr 1986
TX Iota	McMurry University, Abilene	25 Apr 1987
PA Nu	Ursinus College, Collegeville	28 Apr 1987
VA Gamma	Liberty University, Lynchburg	30 Apr 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 Apr 1990
CO Delta	Mesa State College, Grand Junction	27 Apr 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 Apr 1991
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 Mar 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993
MS Epsilon	Delta State University, Cleveland	19 Nov 1994
PA Omicron	University of Pittsburgh at Johnstown, Johnstown	10 Apr 1997
MI Delta	Hillsdale College, Hillsdale	30 Apr 1997
MI Epsilon	Kettering University, Flint	28 Mar 1998
MO Mu	Harris-Stowe College, St. Louis	25 Apr 1998
GA Beta	Georgia College and State University, Milledgeville	25 Apr 1998
AL Eta	University of West Alabama, Livingston	4 May 1998
PA Pi	Slippery Rock University, Slippery Rock	19 Apr 1999
TX Lambda	Trinity University, San Antonio	22 Nov 1999
GA Gamma	Piedmont College, Demorest	7 Apr 2000
LA Delta	University of Louisiana, Monroe	11 Feb 2001
GA Delta	Berry College, Mount Berry	21 Apr 2001
TX Mu	Schreiner University, Kerrville	28 Apr 2001
CA Epsilon	California Baptist University, Riverside	21 Apr 2003
PA Rho	Thiel College, Greenville	13 Feb 2004
VA Delta	Marymount University, Arlington	26 Mar 2004
NY Omicron	St. Joseph's College, Patchogue	1 May 2004
IL Iota	Lewis University, Romeoville	26 Feb 2005
WV Beta	Wheeling Jesuit University, Wheeling	11 Mar 2005
SC Epsilon	Francis Marion University, Florence	18 Mar 2005
PA Sigma	Lycoming College, Williamsport	1 Apr 2005
MO Nu	Columbia College, Columbia	29 Apr 2005
MD Epsilon	Stevenson University, Stevenson	3 Dec 2005
NJ Delta	Centenary College, Hackettstown	1 Dec 2006
NY Pi	Mount Saint Mary College, Newburgh	20 Mar 2007
OK Epsilon	Oklahoma Christian University, Oklahoma City	20 Apr 2007
HA Alpha	Hawaii Pacific University, Waipahu	22 Oct 2007
NC Epsilon	North Carolina Wesleyan College, Rocky Mount	24 Mar 2008
NY Rho	Molloy College, Rockville Center	21 Apr 2009
NC Zeta	Catawba College, Salisbury	17 Sep 2009
RI Alpha	Roger Williams University, Bristol	13 Nov 2009
NJ Epsilon	New Jersey City University, Jersey City	22 Feb 2010
NC Eta	Johnson C. Smith University, Charlotte	18 Mar 2010
AL Theta	Jacksonville State University, Jacksonville	29 Mar 2010
GA Epsilon	Wesleyan College, Macon	30 Mar 2010
FL Gamma	Southeastern University, Lakeland	31 Mar 2010
MA Beta	Stonehill College, Easton	8 Apr 2011
AR Beta	Henderson State University, Arkadelphia	10 Oct 2011
PA Tau	DeSales University, Center Valley	29 Apr 2012
TN Zeta	Lee University, Cleveland	5 Nov 2012
RI Beta	Bryant University, Smithfield	3 Apr 2013
SD Beta	Black Hills State University, Spearfish	20 Sept 2013
FL Delta	Embry-Riddle Aeronautical University, Daytona Beach	22 Apr 2014
IA Epsilon	Central College, Pella	30 Apr 2014
CA Eta	Fresno Pacific University, Fresno	24 Mar 2015
OH Theta	Capital University, Bexley	24 Apr 2015
GA Zeta	Georgia Gwinnett College, Lawrenceville	28 Apr 2015
MO Xi	William Woods University, Fulton	17 Feb 2016
IL Kappa	Aurora University, Aurora	3 May 2016
GA Eta	Atlanta Metropolitan University, Atlanta	1 Jan 2017
CT Gamma	Central Connecticut University, New Britan	24 Mar 2017
KS Eta	Sterling College, Sterling	30 Nov 2017
NY Sigma	College of Mount Saint Vincent, The Bronx	4 Apr 2018
PA Upsilon	Seton Hill University, Greensburg	5 May 2018

KY Gamma	Bellarmine University, Louisville	23 Apr 2019
MO Omicron	Rockhurst University, Kansas City	13 Nov 2020
AK Gamma	Harding University, Searcy	27 Apr 2021
GA Theta	College of Coastal Georgia, Brunswick	22 Oct 2021
CA Theta	William Jessup University, Rocklin	17 Oct 2022